Introduction to rigid supersymmetry

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Outline I

Outline II

3. Superspace and superfields
   - Supersymmetry represented on fields
   - Superspace and superfields
   - Building blocks of the MSSM

4. Supersymmetric actions
   - $\mathcal{N} = 1$ Matter actions
   - $\mathcal{N} = 1$ Super Yang-Mills actions
   - $\mathcal{N} = 1$ Gauge-Matter coupling
   - $\mathcal{N} = 1$ General Lagrangian
   - Extended supersymmetry

The supersymmetry algebra

Most general symmetry

Theorem (Coleman-Mandula)

If G is a connected symmetry group of the S matrix, G is necessarily locally isomorphic to the direct product of an internal symmetry group and the Poincaré group. The internal symmetry group must be of the form of a semisimple group with additional U(1) factors.

Corollary

The most general symmetries of the S-matrix will obey the algebra

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\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} \\
[M_{\mu\nu}, P_\rho] &= -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu \\
[B_l, B_m] &= i f_{lm}{}^n B_n \\
[P_\mu, B_l] &= 0 \\
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Supersymmetry is a Lie...Algebra

Theorem (Haag–Łopuszański–Sohnius)

The Coleman-Mandula theorem can be furthermore generalised by inserting fermionic generators in the theory. Supersymmetry rises.

Definition (Graded Lie algebras)

A graded Lie algebra of grade n, is a vector space L, which can be decomposed as the direct sum of L_i vector spaces:

$$L = \bigoplus_{i=0}^{n} L_i, \quad \text{with } n = 1, 2, 3, ...$$

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Graded Lie algebra properties

Definition

We define the product $[\ ,\ \}: L \times L \to L$ with the following properties:
Grading: For all $x_i \in L_i$

$$[x_i, x_j\} \in L_{i+j\ mod(n+1)}$$

which is the property that makes L a graded algebra
Supersymmetrization: For all $x_i \in L_i,\ x_j \in L_j$

$$[x_i, x_j\} = -(-1)^{ij}\,[x_j, x_i\}$$

where for even ij the product is the commutator, whereas for odd ij the product is the anti-commutator
Generalized Jacobi identity: For all $x_i \in L_i,\ x_j \in L_j,\ x_k \in L_k$

$$(-1)^{ik}\left[x_i, [x_j, x_k\}\right\} + (-1)^{ij}\left[x_j, [x_k, x_i\}\right\} + (-1)^{jk}\left[x_k, [x_i, x_j\}\right\} = 0$$

Supersymmetry is a Lie...Algebra

Definition

We address the SUSY algebra to be a graded Lie algebra of $n = 1$

$$L = L_0 \oplus L_1$$

where L_0 can be recognised as the Poincaré algebra. L_1 is the vector space spanned by $\left(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\right)$ with $I = 1, ..., N$.

The remaining products of the SUSY algebra can be constructed by two means

- The way higher irreducible representations of the Lorentz group can be obtained by decomposition of direct products

$$(j_1, j_2) \otimes (j_1', j_2') = (j_1 + j_1', j_2 + j_2') \oplus (j_1 + j_1' - 1, j_2 + j_2') \oplus ...$$

- The Lorentz invariance imposes the conditions on coefficients: they can be constructed only from the invariant tensors of Lorentz and $SL(2, \mathbb{C})$, i.e., $\eta_{\mu\nu}, \epsilon_{\alpha_\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}, (\sigma_\mu)_{\alpha\dot{\alpha}}, (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}$

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Supersymmetry algebra

1. $\left[M_{\mu\nu}, Q^I_\alpha\right] = i\left(\sigma_{\mu\nu}\right)_\alpha{}^\beta Q^I_\beta \implies \left[J_3, Q^I_1\right] = \frac{1}{2}Q^I_1, \left[J_3, Q^I_2\right] = -\frac{1}{2}Q^I_2$
2. $\left[M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}\right] = i\left(\bar{\sigma}_{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}}$
3. $\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2\sigma^\mu{}_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \implies$ **Two susy xfm's apply a translation**
4. $\left[P_\mu, Q^I_\alpha\right] = 0 \implies$ **Mass degeneracy of susy states**
5. $\left[P_\mu, \bar{Q}^I_{\dot{\alpha}}\right] = 0$
6. $\left\{Q^I_\alpha, Q^J_\beta\right\} = \epsilon_{\alpha\beta} Z^{IJ} \implies$ **Provides central charges**
7. $\left\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\right\} = \epsilon_{\dot{\alpha}\dot{\beta}} \left(Z^{IJ}\right)^*$
8. $\left[Q^I_\alpha, B_l\right] = \left(b_l\right)^I{}_J Q^J{}_\alpha \implies$ **R-symmetry provided by $U(N)_R$**
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Representations of the SUSY algebra

Poincaré Casimirs

One-particle states are simultaneous eigenstates of the Casimir operators of the Poincaré group (P^2, W^2), where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$. It is necessary to distinguish between massive and massless particles

Massive particles (labelled by mass and spin)

- $p^\mu = \left(m, \vec{0}\right)$, eigenstate of P^μ
- $p^2 = m^2$, eigenvalues of P^2
- $W^2 = -m^2 \vec{J}^2$, with eigenvalues of $-m^2 j(j+1)$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Massless particles (labelled by energy and helicity)

- $p^\mu = (E, 0, 0, E)$, eigenstate of P^μ
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- $w^\mu = \lambda p^\mu$, where $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$ the helicity

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SUSY particles as a collection of S.M. particles

Any irrep of the SUSY algebra can be decomposed into an irrep of the Poincaré algebra, since Poincaré is a subalgebra of SUSY. If a particle is an irrep of the Poincaré algebra, then a super-particle is an irrep of the SUSY-algebra.

Corollary

SUSY particle $\equiv$ *multiplet of S.M. particles*

Degeneracy of mass between particles of different spin

P^2 is a Casimir for both Poincaré and SUSY algebra, whereas W^2 is not a SUSY Casimir $\implies$ irreps have the same **mass** as Poincaré but not the same **spin**.

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Physical properties of SUSY states

1. In a supersymmetric theory the energy P_0, of *any* state is always positive $\implies 0 \leq \langle\phi\,|P_0|\,\phi\rangle$

Unitarity of the algebra

In other words, $Q\bar{Q} \sim P_\mu$ being hermitian, ensures the algebra to be unitary, which is a demand in effect.

Example

Particles such as tachyons, are not predicted in supersymmetry, or in general, particles with negative energy norms. Existence of such kind of particles would violate unitarity.

2. The fermionic and bosonic degrees of freedom, residing in a supermultiplet, must be equal, i.e. $n_B = n_F$

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Massless supermultiplets

- Phenomenologically speaking, these multiplets exhibit the most relevance in accordance to the SM, since SM particles are at their majority massless, only acquiring mass from the Higgs. Using

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\right\} = 2\sigma^\mu_{\alpha\dot\beta} p_\mu \delta^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot\beta} \delta^{IJ}$$

one acquires that $Q^I_1 = \bar{Q}^I_1 = 0$ when acted on states.

- Also using

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Central charges vanishes for massless representations of any $\mathcal{N}$.

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Massless supermultiplets operators

Definition (operator algebra)

We define an equivalence of creation and annihilation operators where $a_I^\dagger \equiv \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot{2}}$ will be raising the spin by one half, whereas $a_I \equiv \frac{1}{\sqrt{4E}} Q^I_2$ will be lowering it, such that:

$$\left\{a_I, a_J^\dagger\right\} = \delta^{IJ}, \qquad \{a_I, a_J\} = \left\{a_I^\dagger, a_J^\dagger\right\} = 0$$

Definition (Clifford vacuum)

We define the lower state of helicity as a Clifford vacuum $|\lambda_0\rangle$, for which $a_I |\lambda_0\rangle = 0$.

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Costruction of SUSY states

Getting from the Clifford vaccum to higher helicity states:

$$a_I^\dagger \left|\lambda_0\right\rangle = \left|\lambda_0 + \frac{1}{2}\right\rangle \quad \Rightarrow \quad \text{n states}$$

$$a_I^\dagger a_J^\dagger \left|\lambda_0\right\rangle = \left|\lambda_0 + 1\right\rangle \quad \Rightarrow \quad \binom{n}{2} \text{ states}$$

$$\vdots$$

$$a_{I_1}^\dagger a_{I_2}^\dagger ... a_{I_n}^\dagger \left|\lambda_0\right\rangle = \left|\lambda_0 + \frac{n}{2}\right\rangle \quad \Rightarrow \quad \text{1 state}$$

Because $a^\dagger$ anticommute, the indices of the left side are antisymmetric in their exchange. Since, $n \leq N$ the number of states for a given n are $\binom{N}{n}$. Summing over all n, provides the dimension of the representation (i.e. the number of states).

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N = \underbrace{2^{N-1}}_{number\ of\ bosonic\ states} + \underbrace{2^{N-1}}_{number\ of\ fermionic\ states}$$

$\mathcal{N} = 1$ massless

- The **matter multiplet** can be constructed by acting with the one generator on the Clifford vacuum $|\lambda_0 = 0\rangle$ and adding the CPT conjugate.

$$\left(0, +\frac{1}{2}\right) \underset{CPT}{\oplus} \left(-\frac{1}{2}, 0\right)$$

- The **gauge multiplet** can be constructed likewise,

$$\left(+\frac{1}{2}, +1\right) \underset{CPT}{\oplus} \left(-1, -\frac{1}{2}\right)$$

  - 1 Weyl fermion (2 dof)
  - 1 massless vector (2 dof)

Local supersymmetry

Continuing with the same logic, one would stumble across the gravitino and graviton multiplets. In general, gravity can by added by hand for $N \leq 4$ SUSY but *cannot* be avoided for $N > 4$.

$\mathcal{N} = 1$ massless

- The **matter multiplet** can be constructed by acting with the one generator on the Clifford vacuum $|\lambda_0 = 0\rangle$ and adding the CPT conjugate.

$$\left(0, +\frac{1}{2}\right) \underset{CPT}{\oplus} \left(-\frac{1}{2}, 0\right)$$

- The **gauge multiplet** can be constructed likewise,

$$\left(+\frac{1}{2}, +1\right) \underset{CPT}{\oplus} \left(-1, -\frac{1}{2}\right)$$

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$\mathcal{N} = 2$ massless

- The **matter multiplet** can be constructed

$$\left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right) \underset{CPT}{\oplus} \left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right)$$

  - 2 complex scalars (2 dof each)
  - 2 Weyl fermions (2 dof each)

R-symmetry

The CPT conjugate is added due to the demand of the R-symmetry, for the complex scalar to transform as a doublet under $SU(2)_R$.

- The **gauge multiplet** reads

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$\mathcal{N} = 4$ massless

- No matter multiplet can be accommodated to $\mathcal{N} = 4$ supersymmetry, since it is impossible *not* to exceed spin $\frac{1}{2}$. The **vector** (or gauge) multiplet reads

$$\left(-1,\ \underline{4} \times -\frac{1}{2},\ \underline{6} \times 0,\ \underline{4} \times \frac{1}{2},\ +1\right)$$

The degrees of freedom correspond to that of

  - 1 vector with helicity ± 1 (2 dof)
  - 4 Weyl fermions (8 dof)
  - 3 complex scalars (6 dof)

No problem regarding R-symmetry is present here.

Massive supermultiplets

- Using the massive rest frame $p_\mu = (m, 0, 0, 0)$, we get on states that

$$\left\{ Q^I_\alpha, \bar{Q}^J_{\dot{\beta}} \right\} = 2m\delta_{\alpha\dot{\beta}}\delta^{IJ}$$

  where both of the components of Q and $\bar{Q}$ are non zero, contrary to the massless case. **The dimension of the representations is** 2^{2N}, since there are $2N$ generators. Starting from a fixed Clifford vacuum $|j_0, j_3\rangle$ we will acquire the remaining states of the representation. Also, the Clifford vacuum will have a degeneracy of $2j+1$ since j_3 ranges from $-j$ to $+j$, and is denoted simply as $|j_0\rangle$.
- **Central charges** are generally *non*-vanishing (except for the $\mathcal{N} = 1$ case).

$\mathcal{N} = 1$ massive

Definition (operator algebra for massive $\mathcal{N} = 1$)

We define a set of annihilation and creation operators respectively

$$a_{1,2} \equiv \frac{1}{\sqrt{2m}} Q_{1,2} \qquad a^{\dagger}_{1,2} \equiv \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{1},\dot{2}}$$

which will satisfy the relations

$$\{a, a\} = \left\{a^{\dagger}, a^{\dagger}\right\} = 0, \qquad \left\{a_{\alpha}, a^{\dagger}_{\beta}\right\} = \delta_{\alpha\beta}$$

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We define a Clifford vacuum for which

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$\mathcal{N} = 1$ massive

- Setting $j_0 = 0$ we get the **matter** multiplet of $\mathcal{N} = 1$.

$$\left(-\frac{1}{2}, 0, 0', +\frac{1}{2}\right)$$

- Similarly for the **gauge** multiplet

$$\left(-1, -\frac{1}{2}, -\frac{1}{2}', \underline{2} \times 0, +\frac{1}{2}, +\frac{1}{2}', +1\right)$$

  - one massive vector (-1,0,1), (3 dof)
  - one massive real scalar, (1dof)
  - one massive Dirac fermion, (4dof)

Higgs mechanism

The degrees of freedom are these of gauge+matter of massless $\mathcal{N} = 1$. This is a fortunate consistency since in nature we do not want our vector particles to be massive, rather to acquire mass through a Higgs mechanism.

Superspace and superfields

Supersymmetry represented on fields

Transition from on-shell $\Longrightarrow$ off-shell representations

- The Clifford vacuum is x^μ dependable, i.e. $\phi \Longrightarrow \phi(x)$
- Creation/annihilation operators are represented by differential operators,

$$\left[\bar{Q}_{\dot{\alpha}}, \phi(x)\right] = 0$$

where ϕ is a complex scalar.

Acting with the creation operators and using the generalised Jacobi, the Wess-Zumino multiplet of fields, reads

$$(\phi(x), \psi_\alpha(x), F(x))$$

Off-shell $\Longrightarrow$ on-shell when e.o.m. are imposed

$F(x)$ is a auxiliary field, carrying no dynamical d.o.f. $\Longrightarrow$ vanishes on-shell.

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Superspace

Superspace

The space where supersymmetry invariance manifests naturally for which superfields are naturally defined.

Introduce a new set of coordinates to parametrize the new coset and rewrite the odd part of supersymmetry as a Lie algebra. That can be achieved by introducing the Grassmann numbers θ_α and $\bar{\theta}_{\dot{\alpha}}$

$$Q \to \theta Q, \qquad \bar{Q} \to \bar{\theta}\bar{Q}$$

for which a SuperPoincaré element reads

$$G(x, \theta, \bar{\theta}, \omega) = \exp\left(ixP + i\theta Q + i\bar{\theta}\bar{Q} + \frac{1}{2}i\omega M\right)$$

Definition (Superspace as a coset)

$$\mathcal{M}_{1,3} = \frac{ISO(1,3)}{SO(1,3)} \implies \mathcal{M}_{4|4} = \frac{\text{SuperPoincaré group}}{\text{Lorentz group}}$$

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Most generic superfield

The basic idea is transitioning from

$$\text{space} \longrightarrow \text{superspace}$$

$$\text{fields} \longrightarrow \text{superfields}$$

$$\phi(x), \psi_\alpha(x) \longrightarrow \Phi(x, \theta, \bar{\theta})$$

The most generic scalar superfield can be expanded in terms of $\theta, \bar{\theta}$ as

$$\begin{aligned} Y(x,\theta,\bar{\theta}) =& f(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) \\ &+ \theta\sigma^\mu\bar{\theta}\upsilon_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \end{aligned}$$

a reducible rep of SUSY algebra.

Definition (SUSY generators as differential operators)

Closing the SUSY algebra are defined,

$$Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu$$

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SUSY transformations on fields and the covariant derivative

SUSY transformation on fields

The variation of a superfield with respect to SUSY translational parameters $\epsilon, \bar{\epsilon}$ is given

$$\begin{aligned}\delta_{\epsilon\bar{\epsilon}} Y(x,\theta,\bar{\theta}) &= Y(x+\delta x, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) - Y(x,\theta,\bar{\theta}) \\ &= i\left(\epsilon Q + \bar{\epsilon}\bar{Q}\right) Y(x,\theta,\bar{\theta})\end{aligned}$$

- If S_1 and S_2 are superfields then so is the product $S_1 S_2$
- Linear combinations of superfields are also superfields.
- If Y is a superfield then $\partial_\mu Y$ is a superfield but $\partial_\alpha Y$ is not, since $\partial_\alpha, \bar{\partial}_{\dot{\alpha}}$ does not commute with SUSY transformations.For that reason we need to introduce a set of supersymmetric *covariant* derivatives, namely,

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \qquad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu$$

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Chiral supefields

- If a superfield of the form $\bar{D}_{\dot{\alpha}}\Phi$ is constrained by

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

then Φ is called a *chiral* superfield or $\mathcal{X}_{sf}$.

- If a superfield of the form $D_\alpha \Psi$ is constrained by

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- If Φ is a $\mathcal{X}_{sf}$ then $\Phi^\dagger$ is an anti-$\mathcal{X}_{sf}$

Definition (Chiral superfield)

Utilizing the above, the most general chiral superfield can be expressed,

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \theta\theta F(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta}$$
$$- \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x)$$

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SUSY transformations on matter fields

SUSY transforming the chiral superfield we recognise the variations of the fields to be

$$\begin{aligned}\delta F =& i\sqrt{2}\partial_\mu \psi \sigma^\mu \bar{\epsilon} \\ \delta\psi_\alpha =& -\sqrt{2}\epsilon_\alpha F + i\sqrt{2}\partial_\mu \phi (\sigma^\mu \bar{\epsilon})_\alpha \\ \delta\phi =& \sqrt{2}\psi\epsilon\end{aligned}$$

Fields are transforming into their partners under SUSY transformations.
The algebra closes off-shell, hence quantum effects do not break supersymmetry,

Example (scalar field)

The commutator of two susy transformations is itself a symmetry transformation

$$[\delta_2, \delta_1]\phi = -2i\left(\epsilon_2\sigma^\mu\bar{\epsilon}_1 - \epsilon_1\sigma^\mu\bar{\epsilon}_2\right)\partial_\mu\phi$$

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If a superfield V is constrained by the reality condition $V = V^\dagger$, then V is called a *real* or *vector* superfield.

Using a supergauge transformation of the form

$$V \to V + \Phi + \bar{\Phi}$$

we derive the vector superfield in the Wess-Zumino gauge,

$$V_{WZ}(x, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta}\upsilon_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)$$

- Notice that

$$V_{WZ}^2(x, \theta, \bar{\theta}) = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\upsilon^2(x) \quad V_{WZ}^n = 0, \text{ for } n \geq 3$$

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Supersymmetric actions

Supersymmetric actions

Power of the superfields/superspace formalism

The action must be invariant under SUSY transformations. Written in full superspace the following integral serves as the SUSY action,

$$S = \int d^4x \underbrace{d^2\theta \, d^2\bar{\theta} \, \mathcal{A}(x, \theta, \bar{\theta})}_{\mathcal{L}}$$

and it is automatically SUSY invariant if $\mathcal{A}(x, \theta, \bar{\theta})$ is a superfield.

Properties of the Lagrangian density:

- It must be real and scalar
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$\mathcal{N} = 1$ Matter actions

A natural superfield candidate forms the **matter** Lagrangian

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providing the kinetic terms of the theory.
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$$\mathcal{L}_0 = \bar{\Phi}\Phi|_{\theta\theta\bar{\theta}\bar{\theta}} = \underbrace{\frac{i}{2}\left(\partial_\mu\psi\sigma^\mu\bar{\psi} - \psi\sigma^\mu\partial_\mu\bar{\psi}\right)}_{\mathcal{L}_{fermion}} + \underbrace{\partial_\mu\phi\partial^\mu\bar{\phi}}_{\mathcal{L}_{scalar}} + \underbrace{\bar{F}F}_{\mathcal{L}_{auxiliary}}$$

providing the kinetic terms of the theory.
The equations of motion read,

$$\begin{aligned} \Box\phi(x) &= 0 \\ \bar{\sigma}^\mu\partial_\mu\psi(x) &= 0 \\ F(x) &= 0 \end{aligned}$$

Most general free Lagrangian

Consider a Lagrangian of the form

$$\int d^2\theta \; d^2\bar{\theta} \; K(\Phi, \bar{\Phi})$$

Properties to be met for this to be a Lagrangian

- $K(\Phi, \bar{\Phi})$ should be a superfield
- $K(\Phi, \bar{\Phi})$ should be real and scalar
- $K(\Phi, \bar{\Phi})$ should be of mass dimension two, $[K] = 2$
- $K(\Phi, \bar{\Phi})$ should not be a function of $D\Phi, \bar{D}\bar{\Phi}$ (because we would end up with a higher derivative theory and we don't want that for a minimal supersymmetric model).

A function $K(\Phi, \bar{\Phi})$ that is consistent with the above properties, is of the form,

$$K(\Phi, \bar{\Phi}) = \sum_{n,m=1}^{\infty} c_{nm} \bar{\Phi}^m \Phi^n, \quad c_{mn} = c^*_{nm}$$

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Most general free Lagrangian

Demanding renormalizability

For a renormalizable theory

$$[c] \geq 0 \implies [m] + [n] \leq 2 \implies [m] = [n] = 1$$

but this is nothing more than

$$K(\Phi, \bar{\Phi}) = \bar{\Phi}\Phi$$

which is the most general Kähler potential for a renormalizable theory.

Definition (F-terms, D-terms)

An integral in superspace can always been expressed in half-superspace but the converse is not true. Terms that are able to do so, are called D-terms and terms that are unable F-terms.

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Interactive Lagrangian-Superpotential

Adding interactions could be achieved by examining the F-terms. Let us define an integral of the form

$$\int d^4x\, d^2\theta\, W$$

For the integral to be non-vanishing, W must be a $\mathcal{X}_{sf}$ meaning that $W = W(\Phi)$ a holomorphic function of Φ.

Definition (Superpotential)

$W(\Phi)$ is called the *superpotential* and is expressed as follows,

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$\mathcal{N} = 1$ Matter Lagrangian

Assembling everything, the $\mathcal{N} = 1$ Matter Lagrangian is expressed,

$$\mathcal{L}_{matter} = \underbrace{\int d^2\theta \, d^2\bar{\theta} \, \bar{\Phi}\Phi}_{\mathcal{L}_{kin}} + \underbrace{\int d^2\theta \, W(\Phi) + \int d^2\bar{\theta} \, \overline{W}(\bar{\Phi})}_{\mathcal{L}_{int}}$$

$$= \frac{i}{2} \left(\partial_\mu \psi \sigma^\mu \bar{\psi} - \psi \sigma^\mu \partial_\mu \bar{\psi} \right) + \partial_\mu \phi \partial^\mu \bar{\phi} + \bar{F} F$$

$$- \frac{\partial W}{\partial \phi} F - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \psi\psi + h.c.$$

Definition ($\mathcal{N} = 1$ on-shell matter Lagrangian)

which is on-shell recasted as

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An application of the matter Lagrangian

Example (The renowned Wess-Zumino model)

Using the specific superpotential

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3$$

we end up with the on-shell Lagrangian

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- Mass degeneracy for bosons and fermions
- Same coupling constant for both different kind of interactions i.e. quartic and Yukawa-like $\Longrightarrow$ (solution of Hierarchy problem)
- In a S.M.-like theory, cubic renormalizable interactions still exist for massless theories.

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$\mathcal{N} = 1$ Abelian Super Yang-Mills

Definition (Abelian field strength)

Consider a $U(1)$ theory for which we define the Abelian field-strength

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \qquad \overline{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V$$

where $W/\overline{W}$ are $\mathcal{X}_{sf}$/anti-$\mathcal{X}_{sf}$ superfields respectively. They are also invariant under supergauge transformations.

Explicitly (using the Wess-Zumino gauge), the **gaugino** superfield can be written as

$$W_\alpha = i\lambda_\alpha - \theta_\alpha D - i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} - \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha$$

Demanding for the requested properties of the Lagrangian to be fulfilled

$$\mathcal{L}_{gauge} = \int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar{\theta}\, \overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}$$
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For a non-abelian gauge group, let

$$V \to V_\alpha T^\alpha, \qquad \alpha = 1, ..., \dim G$$

where G the gauge group. The non-abelian generalisation of the super-gauge transformation is written,

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Perform a rescale on the gauge superfield $V \to 2gV$ for the coupling constant to appear, we get the super Yang-Mills Lagrangian

$$\mathcal{L}_{SYM} = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2}D^2\right] + \frac{\theta_{YM}}{32\pi^2} g^2 \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

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$\mathcal{N} = 1$ gauge-matter coupling

Consider a chiral superfield Φ, transforming in some representation R of the gauge group G, $T^a \to (T^a_R)^i{}_j$ for $i, j = 1, 2, ..., \dim R$. Under a gauge transformation,

$$\Phi \to e^{i\Lambda}\Phi, \qquad \Lambda = \Lambda_a T^a, \qquad T^a = T^{a\dagger}$$

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Gauge invariance of the matter terms

Requesting gauge invariance, couples the Kähler potential to be expressed

$$K(\Phi, \bar{\Phi}) = \bar{\Phi} e^V \Phi$$

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Fayet-Iliopoulos term

Superpotential

Notice that is someone wishes to add the superpotential, then it should be fixed to also be gauge invariant $\Longrightarrow$ the coupling constant should contain the invariant tensor of the gauge group.

Definition (Fayet-Iliopoulos term)

When the gauge group contains $U(1)$ factors,

$$\mathcal{L}_{FI} = \sum_A \xi_A \int d^2\theta \, d^2\bar{\theta} V^A = \frac{1}{2} \sum_A \xi_A D^A$$

which is susy and gauge invariant.

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$\mathcal{N}=1$ Most general renormalizable susy Lagrangian

$$
\begin{aligned}
&\mathcal{L}^{\mathcal{N}=1}_{SYM} + \mathcal{L}^{\mathcal{N}=1}_{matter} + \mathcal{L}^{\mathcal{N}=1}_{FI} = \\
=&\frac{1}{32\pi}\,\mathrm{Im}\left(\tau\int d^2\theta\,\mathrm{Tr}\,W^\alpha W_\alpha\right) + 2g\sum_A \xi_A \int d^2\theta d^2\bar{\theta} V^A + \\
&+\int d^2\theta d^2\bar{\theta}\bar{\Phi}^i e^{2gV}\Phi_i + \int d^2\theta W(\Phi^i) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}^i) \\
=&\underbrace{\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}D^2\right] + \frac{\theta_{YM}}{32\pi^2}g^2\,\mathrm{Tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu}}_{\mathcal{L}_{SYM}} \\
&+\underbrace{\left(\overline{D_\mu\phi^i}\right)D^\mu\phi_i - i\psi^i\sigma^\mu D_\mu\bar{\psi}_i + \bar{F}^i F_i + i\sqrt{2}g\bar{\phi}^i\lambda\psi_i - i\sqrt{2}g\bar{\psi}^i\bar{\lambda}\phi_i + g\bar{\phi}^i D\phi_i}_{\mathcal{L}_{matterD}} \\
&\underbrace{-\frac{\partial W}{\partial\phi^i}F^i - \frac{\partial\bar{W}}{\partial\bar{\phi}_i}\bar{F}_i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\psi^i\psi^j - \frac{1}{2}\frac{\partial^2\bar{W}}{\partial\bar{\phi}_i\partial\bar{\phi}_j}\bar{\psi}_i\bar{\psi}_j}_{\mathcal{L}_{matterF}} + \underbrace{g\sum_A \xi_A D^A}_{\mathcal{L}_{FI}}
\end{aligned}
$$

On-shell $\mathcal{N} = 1$ Lagrangian

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^{\mu}D_{\mu}\bar{\lambda}\right] + \overline{D_{\mu}\phi^{i}}D^{\mu}\phi_{i} - i\psi^{i}\sigma^{\mu}D_{\mu}\bar{\psi}_{i} + i\sqrt{2}g\bar{\phi}^{i}\lambda\psi_{i}$$

$$- i\sqrt{2}g\bar{\psi}^{i}\bar{\lambda}\phi_{i} - \frac{1}{2}\frac{\partial^{2}W}{\partial\phi^{i}\partial\phi^{j}}\psi^{i}\psi^{j} - \frac{1}{2}\frac{\partial^{2}\bar{W}}{\partial\bar{\phi}_{i}\partial\bar{\phi}_{j}}\bar{\psi}_{i}\bar{\psi}_{j} - V(\phi,\bar{\phi})$$

with the scalar potential being,

$$V(\phi,\bar{\phi}) = \frac{\partial W}{\partial\phi^{i}}\frac{\partial\bar{W}}{\partial\bar{\phi}_{i}} + \frac{g^{2}}{2}\sum_{a}\left|\bar{\phi}_{i}\left(T^{a}\right)^{i}_{j}\phi^{j} + \xi^{a}\right|^{2} =$$

$$= \bar{F}F + \frac{1}{2}D^{2}\Big|_{\text{on the solution}} \geq 0$$

$\mathcal{N} = 2$ Extended supersymmetry

Utilizing the $\mathcal{N} = 1$ language, $\mathcal{N} = 2$ super Yang-Mills Lagrangian reads,

$$\mathcal{L}_{SYM}^{\mathcal{N}=2} = \frac{1}{32\pi} \operatorname{Im} \left(\tau \int d^2\theta \, \mathrm{Tr} \, W^\alpha W_\alpha \right) + \int d^2\theta d^2\bar{\theta} \, \mathrm{Tr} \, \bar{\Phi} e^{2gV} \Phi$$

- The superpotential is absent
- All fields now transform in the adjoint of the gauge group
- R-symmetry is larger by a $SU(2)_R$ component

R-symmetry constraints

Bosonic fields transform as scalars under $SU(2)_R$, whereas fermionic fields are doublets. A superpotential would give ψ interactions that are absent for λ, hence would break the R-symmetry.

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$$+\frac{1}{2}D^2 + \bar{F}F + i\sqrt{2}g\bar{\phi}\{\lambda,\psi\} - i\sqrt{2}g\{\bar{\psi},\bar{\lambda}\}\phi + gD[\phi,\bar{\phi}]\Bigg\}$$

- The same logic follows for the $\mathcal{N} = 2$ matter Lagrangian
- $\mathcal{N} = 4$ Lagrangian can be constructed likewise, having only a Y-M Lagrangian part and a larger R-symmetry.

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Open problems

- Is spacetime supersymmetry realized at TeV scale? If so, what is the mechanism of supersymmetry breaking?
- Does supersymmetry stabilize the electroweak scale, preventing high quantum corrections?
- Does the lightest supersymmetric particle (LSP) comprise dark matter?

"Supersymmetry was (and is) a beautiful mathematical idea. The problem with applying supersymmetry is that it is too good for this world."

Frank Wilczek

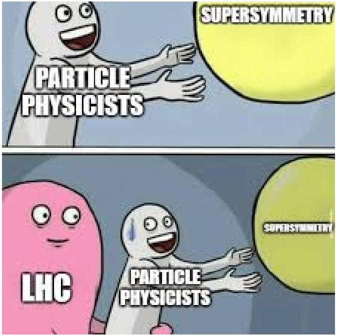

Figure: So close, no *Wess-Zumino* how far