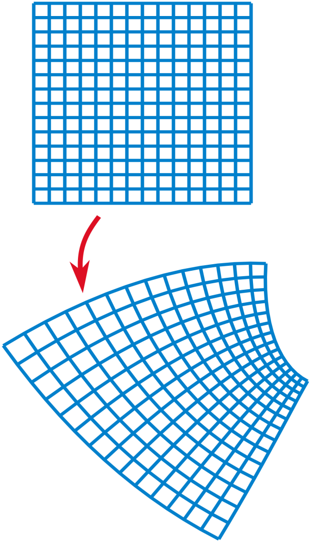
Conformal Field Theory in two dimensions with application on the Ising model

Lazaros Tsaloukidis
Supervisor: Anastasios Petkou

Department of Physics
Aristotle University of Thessaloniki

17-2-2021

Introduction



· A conformal field theory (CFT) is a quantum field theory which is invariant under conformal transformations. This means that the physics of the theory looks the same at all length scales.

· Important applications in various branches of physics, especially in critical phenomena, condensed matter physics and string theory.

· Various systems exhibiting scale invariance are studied with Conformal Field Theory.

· In two dimensions, there is an infinite-dimensional algebra of local conformal transformations, and Conformal Field Theories can sometimes be exactly solved or classified.

· Boundary Conformal Field Theory (bCFT) is of great interest the last years, with active research being performed.

Basics of CFT in 2 dimensions

Conformal transformations and generators

$$x^\mu \to x^\mu + \epsilon^\mu \qquad ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu \qquad g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$$

- $\epsilon^\mu = \alpha^\mu \quad \to \quad x' = x + \alpha$ Translations
- $\epsilon^\mu = \omega^\mu_{\ \nu} x^\nu \quad \to \quad x' = \Lambda x \ \ (\Lambda^\mu_{\ \nu} \in SO(p,q))$ Rotations
- $\epsilon^\mu = \lambda x^\mu \quad \to \quad x' = \lambda x$ Scale Transformations
- $\epsilon^\mu = b^\mu x^2 - 2(b \cdot x) x^\mu \quad \to \quad x' = \frac{x + bx^2}{1 + 2b \cdot x + b^2 x^2}$ Special Conformal Tr.

Generators of Conformal Transformations-Conformal Algebra

$$P_\mu = -i\partial_\mu$$

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

$$D = -i x^\mu \partial_\mu$$

$$K_\mu = -i(2x_\mu \partial^\nu \partial_\nu - x^2 \partial_\mu)$$

Poincare $+$
$$\begin{cases} [D, P_\mu] = iP_\mu \\ [D, K_\mu] = -iK_\mu \\ [K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \\ [K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \end{cases}$$

Conformal algebra in 2 dimensions

$$ds^2 = (dx^0)^2 + (dx^1)^2 = dz d\bar{z} \qquad g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

Witt Algebra

$$[l_m, l_n] = (m-n) l_{m+n} \qquad [\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n} \qquad [l_m, \bar{l}_n] = 0$$

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$$

Virasoro Algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n) \bar{L}_{m+n} + \frac{\bar{c}}{12}(m^3 - m)\delta_{m+n,0}$$

$$[L_m, \bar{L}_n] = 0$$

Primary Fields and Correlation Functions I

Definition of Quasi-Primary Field: $\phi_j(z,\bar{z}) = \left|\frac{\partial z'}{\partial z}\right|^h \left|\frac{\partial \bar{z}'}{\partial \bar{z}}\right|^{\bar{h}} \phi_j(z',\bar{z}') \quad h = \Delta_j/2$

For the two-point function:

1st: Invariance under translations and rotations causes the determinant to be equal to unity and implies that the function is of the form $f(z_1 - z_2)$

2nd: Invariance under scale transformation fixes the function
$f(z_1 - z_2) = \lambda^{\Delta_1+\Delta_2} f(\lambda(z_1 - z_2))$

3rd: Invariance of the two-point function under special conformal transformations essentially implies the invariance under transformations $f(z) = -1/z$

The two-point function vanishes unless the two fields have the same scaling dimension.

Two-point function

$$< \phi_1(z_1), \phi_2(z_2) >= \frac{C_{12}}{(z_1 - z_2)^{2\Delta}} \qquad \Delta_1 = \Delta_2 = \Delta$$

Primary Fields and Correlation Functions II

Three-point function

$$< \phi_1(z_1), \phi_2(z_2), \phi_3(z_3) >= \frac{C_{123}}{(z_1-z_2)^{\Delta-2\Delta_3} \cdot (z_2-z_3)^{\Delta-2\Delta_1} \cdot (z_3-z_1)^{\Delta-2\Delta_2}}, \; \Delta = \sum_i \Delta_i$$

Four-point function

$$< \phi_1(z_1), \phi_2(z_2), \phi_3(z_3), \phi_4(z_4) >= f\left(\frac{z_{12}z_{34}}{z_{13}z_{24}}\right) \prod_{i<j} \frac{1}{z_{ij}^{\Delta_i/2+\Delta_j/2+\Delta/6}}$$

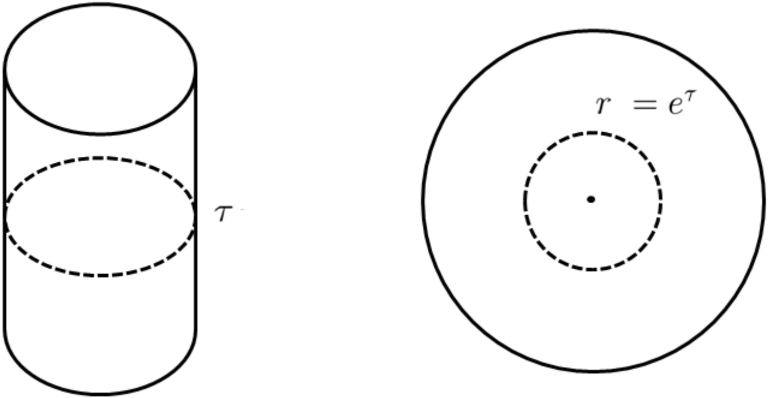
$$z_{ij} = z_1 - z_j$$

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \qquad 1 - x = \frac{z_{14}z_{23}}{z_{13}z_{24}} \qquad \frac{x}{1-x} = \frac{z_{12}z_{34}}{z_{14}z_{23}}$$

$SL(2,C)/Z_2$ global symmetry makes it possible to map any three points $\{z_1, z_2, z_3\}$ at maximum on the Riemann sphere to $\{0, 1, \infty\}$. For the 4-point function we have:

$$z_1 \to \infty, \quad z_2 = 1, \quad z_3 = 0 \quad \text{and} \quad z_4 = x, 1 - x \text{ or } \frac{x}{1-x}$$

Radial Quantization



Euclidean Space coordinates $w, \bar{w} = \tau + i\sigma$ with compactification $\sigma = \sigma + 2\pi$.

Conformal Mapping $z = e^{w} = e^{(\tau + i\sigma)}$: Cylinder $\rightarrow$ Plane

Dilatations Generator: $H = L_0 + \bar{L}_0$ Rotations Generator: $P = i(L_0 - \bar{L}_0)$

Energy Momentum Tensor

Noether Current: $j^\mu_a = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right]\frac{\delta x^\nu}{\delta \epsilon_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\delta F}{\delta \epsilon_a}$ $\quad j_\mu = T_{\mu\nu}\epsilon^\nu$

Stress-Energy Tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi - \eta^{\mu\nu}\mathcal{L} \qquad T^\mu_\mu = 0$$

In the case of curved space-time (dynamical background metric) we generalize to get the Hilbert stress-energy tensor:

$$T^{\mu\nu} = -\frac{4\pi}{\sqrt{|g|}}\frac{\partial S}{\partial g_{\mu\nu}} \qquad \nabla_\mu T^{\mu\nu} = 0 \qquad < T^\mu_\mu >= -\frac{c}{12}R \text{ (Weyl Anomaly)}$$

Conserved Charges

$$Q = \int dx^1 j_0 \quad \rightarrow \quad Q = \frac{1}{2\pi i}\oint [dz T(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\epsilon(\bar{z})]$$

Operator Product Expansion

$$T(z)\phi(w,\overline{w}) = \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w}) + ...$$

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + ...$$

Equations like those above appear quite frequently on CFT and are important, especially the second one (proof that the tensor is conformal field of weight 2).

$$T(z) = \sum_{n\in Z} z^{-n-2}L_n \qquad L_n = \frac{1}{2\pi i}\oint z^{n+1}T(z)dz$$

$$\cdot\ T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12}S(f(z),z)$$

$$\cdot\ S(f(z),z) = \frac{1}{(\partial_z f)^2}\left[(\partial_z f)(\partial_z^3 f) - \frac{3}{2}(\partial_z^2 f)^2\right]$$

By making use of the above we can derive:

Conformal Ward Identity

$$< T(z)\phi_1(w_1,\bar{w}_1)...\phi_N(w_N,\bar{w}_N) >=$$

$$= \sum_{i=1}^{N}\left(\frac{h_i}{(z-w_i)^2} + \frac{\partial_{w_i}}{(z-w_i)}\right) < \phi_1(w_1,\bar{w}_1)...\phi_N(w_N,\bar{w}_N) >$$

Representation theory of the Virasoro algebra

In and Out states of the theory

Laurent expansion of the fields in their respective modes:

$$\left.\begin{aligned}\phi(z,\bar z) &= \sum_{n,\overline{m}\in Z}\frac{1}{z^{n+h}}\frac{1}{\bar z^{\overline{m}+\bar h}}\phi_{n,\overline{m}}\\ \phi^\dagger(z,\bar z) = \phi\left(\frac{1}{\bar z},\frac{1}{z}\right)\frac{1}{\bar z^{2h}}\frac{1}{z^{2\bar h}} &= \sum_{n,\overline{m}\in Z}\bar z^{n-h} z^{\overline{m}-\bar h}\phi_{n,\overline{m}}\end{aligned}\right\} \Rightarrow (\phi_{n,\overline{m}})^\dagger = \phi_{-n,-\overline{m}}$$

In and out states are defined as:

$$|\phi_{in}\rangle = \lim_{z,\bar z\to 0}\phi(z,\bar z)\,|0\rangle = \phi_{-h,-\bar h}\,|0\rangle \qquad \phi_{n,\overline{m}}\,|0\rangle = 0 \quad n>-h,\quad \overline{m}>-\bar h$$

$$\langle\phi_{out}| = \lim_{z,\bar z\to\infty} z^{2h}\bar z^{2\bar h}\,\langle 0|\,\phi(z,\bar z) = \langle 0|\,\phi_{h,\bar h} \quad \langle 0|\,\phi_{n,\overline{m}} = 0 \quad n<h,\quad \overline{m}<\bar h$$

For the stress-energy tensor similarly:

$$T(z)\,|0\rangle = \textstyle\sum_{n\in Z}\frac{L_n}{z^{n+2}}\,|0\rangle \quad L_n\,|0\rangle = 0 \quad n\geq -1 \qquad \langle 0|\,L_n = 0 \qquad n\leq 1$$

Highest-weight states

$\cdot[L_n, \phi(w)] = h(n+1)w^n\phi(w) + w^{n+1}\partial\phi(w)$ $\quad \cdot [L_m, \phi_n] = [m(h-1) - n]\phi_{n+m}$

$$\cdot[L_n, \phi(0)] = 0 \Rightarrow L_n\phi(0)\left|0\right\rangle = L_n\left|h\right\rangle = 0, \quad n > 0$$

Highest weight state

In: $L_0 \left|h\right\rangle = h\left|h\right\rangle$ Out: $\left\langle h\right| L_0 = h\left\langle h\right|$

We check from the above that we have $[L_0, \phi_n] = -n\phi_n$, which for $n = -h$ results in $L_0\left|h\right\rangle = L_0\phi_{-h}\left|0\right\rangle = h\left|h\right\rangle$. This means that the state $\left|h\right\rangle$ is created by the mode ϕ_{-h}, i.e:

$$\left|h\right\rangle = \phi_{-h}\left|0\right\rangle$$

Requiring now that creation operators should create states with positive energy, we conclude that:

- ϕ_n with $n > -h$ are annihilation operators
- ϕ_n with $n \leq -h$ are creation operators

Ward identities

For an arbitrary number of quasi-primary fields we have:

$$\cdot \langle 0| [L_k, \phi_1(z_1)] ... \phi_n(z_n) |0\rangle + ... + \langle 0| \phi_1(z_1) ... [L_k, \phi_n(z_n)] |0\rangle = 0$$

$$\cdot [L_n, \phi(w)] = h(n+1) w^n \phi(w) + w^{n+1} \partial \phi(w)$$

Combining those we get:

$$\sum_{i=1}^{n} \partial_i \langle 0| \phi_1(z_1) ... \phi_n(z_n) |0\rangle = 0$$

$$\sum_{i=1}^{n} (z_i \partial_i + h_i) \langle 0| \phi_1(z_1) ... \phi_n(z_n) |0\rangle = 0$$

$$\sum_{i=1}^{n} (z_i^2 \partial_i + 2 z_i h_i) \langle 0| \phi_1(z_1) ... \phi_n(z_n) |0\rangle = 0$$

Descendant Fields

$$\text{OPE: } T(z)\phi(w) = \frac{1}{(z-w)^2} L_0\phi(w) + \frac{1}{(z-w)} L_{-1}\phi(w) + L_{-2}\phi(w) + ...$$

$$\text{Natural Definition: } \phi^{-n}(w) = (L_{-n}\phi)(w) = \frac{1}{2\pi i}\oint \frac{1}{(z-w)^{n-1}} T(z)\phi(w)dz$$

Each primary field $\phi(z)$ gives rise to an infinite set of descendant fields by taking derivatives ∂^k and taking normal ordered products with T. Together, this set of fields, comprise a representation:

Conformal Family

$$[\phi(z)] := \{\phi, \partial\phi, \partial^2\phi, ..., T\phi, ...\}$$

Under the conformal transformations every member of each conformal family transforms in terms of the representations of the same conformal family.

Descendant Field correlation function

$$< (L_{-n}\phi)(w)\phi_1(w_1)...\phi_k(w_k) >= \mathcal{L}_{-n} < \phi(w)\phi_1(w_1)...\phi_k(w_k) >$$

$$\mathcal{L}_{-n} = \sum_{i=1}^{k} \left[\frac{(n-1)}{(w_i-w)^n} h_i - \frac{1}{(w_i-w)^{n-1}} \partial_{w_i} \right]$$

Verma Module/Null states I

State-Operator Correspondence: Each descendant state could be viewed as the application of a descendant field on the vacuum, i.e.: $L_{-n}\left|h\right\rangle = \phi^{-n}(0)\left|0\right\rangle$

Verma Module

Verma module is the set of states corresponding to the conformal family of a primary field and the Hilbert space can be decomposed into a (possibly infinite) direct sum of irreducible highest weight Verma Modules $V_{h,c}$ of the Virasoro algebra

$$\mathcal{H} = \bigoplus_h V_{h,c}$$

The number of states given per level N are equal to the number $P(N)$ of possible ways of the writing the number N as sum of positive integers.

Null States

Null State is a linear combination of states that vanishes, and the representation of the Virasoro algebra with highest weight $\left|h\right\rangle$ is constructed from the above Verma module by removing all null states (and their descendants).

Verma Module/Null states II

N=2 null state

$$|\chi^{null}\rangle_{N=2} = \left(L_{-2} - \tfrac{3}{2(2h+1)}L_{-1}^2\right)|h\rangle = 0 \quad c = \frac{2h(5-8h)}{2h+1} \quad a = -\frac{3}{2(2h+1)}$$

Level	**Dimension**	**Field**	**State**
0	h	$\phi(z)$	$\|h\rangle$
1	$h+1$	$\partial\phi$	$L_{-1}\|h\rangle$
2	$h+2$	$\partial^2\phi$	$L_{-1}L_{-1}\|h\rangle$
2	$h+2$	$T\phi$	$L_{-2}\|h\rangle$
3	$h+3$	$\partial^3\phi$	$L_{-1}L_{-1}L_{-1}\|h\rangle$
3	$h+3$	$T(\partial\phi)$	$L_{-2}L_{-1}\|h\rangle$
3	$h+3$	$\partial(T\phi)$	$L_{-3}\|h\rangle$
...	...	...	...
N	$h+N$	$P(N)$	$P(N)$

Kač Determinant

Previous process of finding the null states of a Verma Module becomes increasingly difficult. We introduce the Kač Determinant:

$$detM_N(h,c) = a_N \prod_{0<p,q\leq N} (h - h_{p,q}(c))^{P(N-pq)} \quad \langle h| \prod_i L_{k_i} \prod_j L_{-m_j} |h\rangle$$

$$h_{p,q}(m) = \tfrac{[(m+1)p-mq]^2-1}{4m(m+1)}, \quad m = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\tfrac{25-c}{1-c}} \quad \sum_{N=0}^{\infty} P(N)q^N = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Level-2

$$\cdot \langle h| L_2 L_{-2} |h\rangle = 4h + \tfrac{c}{2} \qquad \cdot \langle h| L_1 L_1 L_{-2} |h\rangle = 6h$$

$$\cdot \langle h| L_2 L_{-1} L_{-1} |k\rangle = 6h \qquad \cdot \langle h| L_1 L_1 L_{-1} L_{-1} |h\rangle = 4h + 8h^2$$

$$detM_2(h,c) = det \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} = 32(h-h_{1,1})(h-h_{1,2})(h-h_{2,1}) = 0$$

$$h_{1,1} = 0 \quad h_{1,2} = \tfrac{5-c}{16} - \tfrac{1}{16}\sqrt{(1-c)(25-c)} \quad h_{2,1} = \tfrac{5-c}{16} + \tfrac{1}{16}\sqrt{(1-c)(25-c)}$$

Fusion Rules

Minimal models: special cases of rational CFTs, 2-dimensional conformal field theories with a finite number of conformal primary fields.

$$[\phi_{(p_1,q_1)}] \times [\phi_{(p_2,q_2)}] = \sum_{\substack{p_3=|p_1-p_2|+1 \\ p_1+p_2+p_3=\text{odd}}}^{p_1+p_2-1} \sum_{\substack{q_3=|q_1-q_2|+1 \\ q_1+q_2+q_3=\text{odd}}}^{q_1+q_2-1} [\phi_{(p_3,q_3)}]$$

Kač Table: The closed set of operators that defines the specific minimal model $\mathcal{M}_{m,m+1}$. Each primary operator appears twice in this table, due to the symmetry:

$$h_{p,q} = h_{m-p,m+1-q} \quad \rightarrow \quad \phi_{p,q} = \phi_{m-p,m+1-q}$$

and possess $\mathbb{Z}_2$ charge $(-1)^{l_{p,q}}$, where

$$l_{p,q} = \begin{cases} p-1 & \text{mod } 2 \qquad \text{if p is even} \\ q-1 & \text{mod } 2 \qquad \text{if q is even} \\ p+q & \text{mod } 2 \qquad \text{otherwise} \end{cases}$$

Correlation Functions, Structure Constants and Crossing Symmetry

3-point Functions Structure constants

General expression for the OPE of two primary fields:

$$\phi_i(z,\bar{z})\phi_j(0,0) = \sum_p \sum_{\{n,\bar{n}\}} z^{-h^p_{ij}+N} \bar{z}^{-\bar{h}^p_{ij}+\bar{N}} C^{p(-n,-\bar{n})}_{ij} \phi_p^{(-n,-\bar{n})}(0,0)$$

leading to:

$$\langle h_k, \bar{h}_k | \, \phi_i(z,\bar{z})\phi_j(0,0) \, |0\rangle = z^{-h^p_{ij}} \bar{z}^{-\bar{h}^p_{ij}} C^{k(0,0)}_{ij}$$

General expression for the 3-point function from Conformal Invariance:

$$< \phi_k(z_1,\bar{z}_1)\phi_i(z_2,\bar{z}_2)\phi_j(z_3,\bar{z}_3) >= \frac{C_{kij}}{(z_{12})^{h^j_{ki}}(z_{23})^{h^k_{ij}}(z_{13})^{h^i_{kj}} \times (\bar{z}_{12})^{\bar{h}^j_{ki}}(\bar{z}_{23})^{\bar{h}^k_{ij}}(\bar{z}_{13})^{\bar{h}^i_{kj}}}$$

for $z_1 \to \infty$,$z_2 = z$ and $z_3 = 0$ leads to:

$$< \phi_k(\infty,\infty)\phi_i(z,\bar{z})\phi_j(0,0) >= z^{-h^p_{ij}} \bar{z}^{-\bar{h}^p_{ij}} C_{kij}$$

Comparing both expressions we deduce that:

$$C^{k(0,0)}_{ij} = C_{kij}$$

Descendant Fields Structure constants I

Dual basis: $\langle h_k^{(-n,-\bar{n})}| = \sum_{\{n,\bar{n}\}} a_k^{(-n,-m)} \bar{a}_k^{(-\bar{n},-\overline{m})} \langle h_k, \bar{h}_k| \, L_{m_1} ... L_{m_p} \bar{L}_{\overline{m}_1} ... \bar{L}_{\overline{m}_p}$

3-point function: $< \phi_k L_m \phi_i(z,\bar{z}) \phi_j >= z^{-h_{ij}^k + m} \bar{z}^{\bar{h}_{ij}^k} (m h_i - h_j + h_k) C_{ij}^k$

Combining all the above we have:

BPZ Theorem

$$C_{ij}^{k(-n,-\bar{n})} = C_{kij} \times \beta_{ij}^{k(-n)} \times \bar{\beta}_{ij}^{k(-\bar{n})}$$

where

$$\beta_{ij}^{k(-n)} = \sum_{\{n\}} a_k^{(-n,-m)} b_{ij}^{k(-m)} \qquad \bar{\beta}_{ij}^{k(-\bar{n})} = \sum_{\{\bar{n}\}} \bar{a}_k^{(-\bar{n},-\overline{m})} \bar{b}_{ij}^{k(-\overline{m})}$$

• Coefficients are $\beta_{ij}^{k(-n)}$,$\bar{\beta}_{ij}^{k(-\bar{n})}$ "purely kinematic" i.e. they are completely determined by the Virasoro commutation rules and the OPEs of $T\phi$.

• Structure constants for primary operators C_{ijk}, are 'dynamic", i.e. they are determined non-trivially from the full structure of the operator algebra.

Descendant Fields Structure constants II

Identity

$$L_n \left|X^k_{ij}(N+n,\bar{N})\right\rangle = (nh_i - h_j + h_k + N)\left|X^k_{ij}(N,\bar{N})\right\rangle$$

where

$$\left|X^k_{ij}(N,\bar{N})\right\rangle = \sum_{\{n,\bar{n}\}} \beta^{k(-n)}_{ij}\bar{\beta}^{k(-\bar{n})}_{ij} z^N \bar{z}^{\bar{N}} L_{-n_1}...L_{-n_l}\bar{L}_{-\bar{n}_1}...\bar{L}_{-\bar{n}_l}\left|h_k,\bar{h}_k\right\rangle$$

Making use of the above identity along side the Virasoro commutations rules provides the constants for each level.

Example-Level 1

$\cdot \left|X^k_{ij}(0,0)\right\rangle = \left|h_k,\bar{h}_k\right\rangle \quad \cdot \left|X^k_{ij}(1,0)\right\rangle = \beta^{k(-1)}_{ij} L_{-1}\left|h_k,\bar{h}_k\right\rangle \quad \cdot [L_1, L_{-1}] = 2L_0$

$$\left.\begin{aligned} L_1\left|X^k_{ij}(1,0)\right\rangle &= \beta^{k(-1)}_{ij}[L_1,L_{-1}]\left|h_k,\bar{h}_k\right\rangle \\ L_1\left|X^k_{ij}(1,0)\right\rangle &= (h_i-h_j+h_k)\left|X^k_{ij}(0,0)\right\rangle \end{aligned}\right\} \Rightarrow \beta^{k(-1)}_{ij} = \frac{h_i-h_j+h_k}{2h_k}$$

At a given level N there are $P(N)$ coefficients $\beta^{k(-n)}_{ij}$ to be found by bringing $\left|X^k_{ij}(N,\overline{N})\right\rangle$ to level 0 with the help of the Virasoro operators $L_n (n>0)$.

Conformal Blocks and Bootstrap I

Four point function

$$G^{ji}_{kl}(x,\bar{x}) = \langle h_i, \bar{h}_i |\, \phi_j(1,1)\phi_k(x,\bar{x})\, | h_l, \bar{h}_l \rangle \quad x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$$

Decomposition into partial waves formalism

$$G^{ji}_{kl}(x,\bar{x}) = \sum_p C_{pkl} C_{pij} A^{ji}_{kl}(p|x,\bar{x})$$

where

$$A^{ji}_{kl}(p|x,\bar{x}) = \frac{x^{-h^p_{kl}}\bar{x}^{-\bar{h}^p_{kl}}}{C_{pij}} \langle h_i, \bar{h}_i |\, \phi_j(1,1) X^p_{kl}(x,\bar{x}|N,\bar{N})\, |0\rangle = \mathcal{F}^{ji}_{kl}(p|x)\bar{\mathcal{F}}^{ji}_{kl}(p|\bar{x})$$

The conformal blocks represent the element in four-point functions that can be determined from conformal invariance. Dependence on the anharmonic ratios through a series expansion. Fixed by Conformal Invariance.

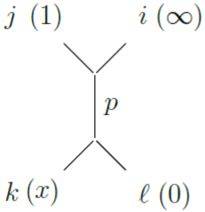
$$G^{ji}_{kl}(x,\bar{x}) = \sum_p C_{pkl} C_{pij} \mathcal{F}^{ji}_{kl}(p|x)\bar{\mathcal{F}}^{ji}_{kl}(p|\bar{x})$$

Conformal Blocks and Bootstrap II

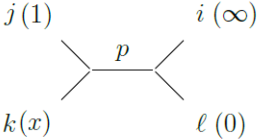
Crossing Symmetry

$$\sum_p C_{pkl} C_{pij} \mathcal{F}^{ji}_{kl}(p|x) \bar{\mathcal{F}}^{ji}_{kl}(p|\bar{x}) = \sum_q C_{qkj} C_{qil} \mathcal{F}^{li}_{kj}(q|1-x) \bar{\mathcal{F}}^{li}_{kj}(q|1-\bar{x})$$

$\mathcal{F}^{ji}_{k\ell}\ (p|x)\ =$



$\mathcal{F}^{ji}_{k\ell}(p|x)\ =$



s-channel: $z_1 \to \infty, z_2 = 1, z_3 = x, z_4 = 0$

t-channel: $z_1 \to \infty, z_2 = 0, z_3 = 1-x, z_4 = 1$

· N^4 constraints to be used for evaluation of $N^3 + N$ parameters.

· Bootstrap approach: Calculation of correlation functions by assuming Crossing Symmetry

· Further constraints taken by making use of the null descendants to find the Conformal Blocks.

Null Vector condition differential equation

As an example we choose the primary field $\phi_{(2,1)}$. We have:

$$\chi^{null}_{(2,1)} = L_{-2}\phi_{(2,1)} - \frac{3}{2(2h_{2,1}+1)}L^2_{-1}\phi_{(2,1)} = 0 \quad h_{2,1} = \frac{5-c}{16} + \frac{1}{16}\sqrt{(1-c)(25-c)}$$

which translates for the correlation function into:

$$\sum_{i=1,2,3}\left[\frac{h_i}{(z_i-z_0)^2} - \frac{1}{(z_i-z_0)}\frac{\partial}{\partial z_i} - \frac{3}{2(2h_{2,1}+1)}\frac{\partial^2}{\partial z_i^2}\right] < \phi_{(2,1)}\prod_{i=1,2,3}\phi_i(z_i,\bar{z}_i) >= 0$$

Taking into account that:

$$\cdot < \phi_{(2,1)}\prod_{i=1,2,3}\phi_i(z_i,\bar{z}_i) >= \prod_{0\leq i<j\leq 3}(z_i-z_j)^{\mu^{ij}}(\bar{z}_i-\bar{z}_j)^{\overline{\mu}^{ij}}f(x,\bar{x})$$

$$\cdot\mu_{ij} = \frac{1}{3}\sum_{k=0}^{3}h_k - h_i - h_j \qquad \cdot\,\overline{\mu}_{ij} = \frac{1}{3}\sum_{k=0}^{3}\bar{h}_k - \bar{h}_i - \bar{h}_j$$

we arrive after a series of transformations and $f'(x) = z^{\mu_{01}}(1-z)^{\mu_{02}}f(x)$ at:

$$\left[\frac{3}{2(2h_{(2,1)}+1)}\frac{d^2}{dx^2} + \frac{2x-1}{x(x-1)}\frac{d}{dx} - \frac{h_1}{x^2} - \frac{h_2}{(x-1)^2} + \frac{h_0+h_1+h_2-h_3}{x(x-1)}\right]f'(x) = 0$$

Free fields on the plane

Free Bosons on the plane

$$\cdot S = \frac{a}{2}\int \partial\phi\bar{\partial}\phi dz d\bar{z} \qquad \cdot\, \partial\bar{\partial}\phi(z,\bar{z}) = 0 \qquad \cdot\, G(z,\bar{z},w,\bar{w}) = -\frac{1}{4\pi a} ln|z-w|^2$$

Currents of the theory: $j(z) = i\partial\phi(z,\bar{z})$ $\quad \bar{j}(\bar{z}) = i\bar{\partial}\phi(z,\bar{z})$

$$< j(z), j(w) >= \frac{1}{4\pi a}\frac{1}{(z-w)^2} \qquad [j_n(z), j_m(w)] = \frac{n}{4\pi a}\delta_{n+m,0}$$

Stress-Energy Tensor and OPEs:

$$\cdot T(z) = -2\pi a \lim_{w\to z}\left[\partial\phi(z,\bar{z})\partial\phi(w,\overline{w}) + \frac{1}{8\pi^2 a^2}\frac{1}{(z-w)^2}\right] = -2\pi a :\partial\phi\partial\phi:$$

$$\cdot T(z)\partial\phi(w,\overline{w}) = \frac{\partial\phi(w,\overline{w})}{(z-w)^2} + \frac{1}{(z-w)}\partial^2\phi(w,\overline{w}) + \ldots$$

$$\cdot T(z)T(w) = \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \ldots$$

Central charge of the free boson is **c**$=1$

Free (Majorana) Fermions on the plane

$$\cdot S = a\int(\overline{\psi}\partial\overline{\psi} + \psi\bar{\partial}\psi)dzd\bar{z} \quad \{\gamma^{\mu},\gamma^{\nu}\} = 2\eta^{\mu\nu}\mathbf{I}_2 \quad \cdot\bar{\partial}\psi = 0 \quad \cdot\partial\bar{\psi} = 0$$

$$\cdot G(z,w) = <\psi(z)\psi(w)> = \frac{1}{2\pi a}\frac{1}{(z-w)} \quad \cdot G(\bar{z},\overline{w}) = <\bar{\psi}(\bar{z})\bar{\psi}(\overline{w})> = \frac{1}{2\pi a}\frac{1}{(\bar{z}-\overline{w})}$$

$$\cdot\{\psi_n(z),\psi_m(w)\} = \frac{1}{2\pi a}\delta_{n+m,0}$$

Stress-Energy Tensor and OPEs:

$$\cdot T(z) = -2\pi T_{zz} = -\frac{1}{2}\pi T^{zz} = -\pi a :\psi\partial\psi:$$

$$\cdot T(z)\psi(w) = \frac{1}{2}\frac{1}{(z-w)^2}\psi(w) + \frac{1}{(z-w)}\partial\psi(w) + \ldots$$

$$\cdot T(z)T(w) = \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \ldots$$

Central charge of the free fermion is **c**$=1/2$

Fermionic Twist fields

$\mathbb{Z}_2$ symmetry, on the Majorana Fermion action results in periodic and antiperiodic boundary conditions on the plane.

$$\psi(w)_{cyl} \to \psi_{cyl}(z) = \left(\frac{dz}{dw}\right)^{1/2} \psi_{pl}(z) = \sqrt{\frac{2\pi z}{L}}\psi_{pl}(z)$$

$$\cdot N.S.(Periodic) : \psi(e^{2\pi i} z) = \psi(z) \quad i\psi(z) = \sum_{n\in\mathbb{Z}+1/2} \psi_n(z) z^{-n-\frac{1}{2}}$$

$$\cdot R.(Antiperiodic) : \psi(e^{2\pi i} z) = -\psi(z) \quad i\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n(z) z^{-n-\frac{1}{2}}$$

Introduction of twist field:

$$\psi(z)\sigma(w) = \frac{\mu(w)}{\sqrt{z-w}} + ... \qquad < \psi(z)\psi(w) >_{(R)} = \frac{1}{4\pi a}\frac{\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}}{z-w}$$

$$\langle\sigma| T(z) |\sigma\rangle = \frac{1}{16z^2} \qquad T(z)|\sigma\rangle = T(z)\sigma(0)|0\rangle = \frac{h_\sigma \sigma(0)}{z^2}|0\rangle + ...$$

The smallest irreducible representation of 2d Clifford algebra consists of two states:

$$(-1)^F |\pm\rangle_R = \pm |\pm\rangle_R \quad \psi_0 |\pm\rangle_R = \frac{1}{\sqrt{4\pi a}} |\pm\rangle_R \quad |+/-\rangle_R = \sigma(0)/\mu(0)|0\rangle_R$$

Bosonization

Using the bosonic field ϕ, we can introduce a *vertex operator*:

$$\mathcal{V}_b(z,\bar{z}) =: e^{ib\phi(z,\bar{z})} : \qquad T(z)\mathcal{V}_b(w,\overline{w}) = \frac{b^2}{8\pi a}\frac{\mathcal{V}_b(w,\overline{w})}{(z-w)^2} + \frac{\partial\mathcal{V}_b(w,\overline{w})}{(z-w)} + \ldots$$

We define now the fermionic fields as:

$$\psi_+ =: e^{i\sqrt{2\pi a}\phi(z)} : \quad \psi_- =: e^{-i\sqrt{2\pi a}\phi(z)} : \quad \psi_\pm = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)$$

From the two relations:

$$\mathcal{V}_b(z)\mathcal{V}_{-b}(w) =: e^{i\sqrt{2\pi a}\phi(z)} :: e^{-i\sqrt{2\pi a}\phi(w)} := \frac{1}{(z-w)} + i\partial\phi(w) + \ldots \quad b = \sqrt{2\pi a}$$

$$\psi_+(z)\psi_-(w) = \frac{1}{(z-w)} + :\psi_+(w)\psi_-(w): + (z-w) < \partial\psi_+(w)\psi_-(w) > + \ldots$$

we extract:

Bosonization Rules

$$:\psi_+(w)\psi_-(w) := i\partial\phi(w) = j(w)$$

$$T_\psi(z) = -\pi a : \psi_1\partial\psi_1 : -\pi a : \psi_2\partial\psi_2 := -\frac{1}{2}T_\phi(z)$$

Application on 2d Ising model

Critical Phenomena-Phase Transition

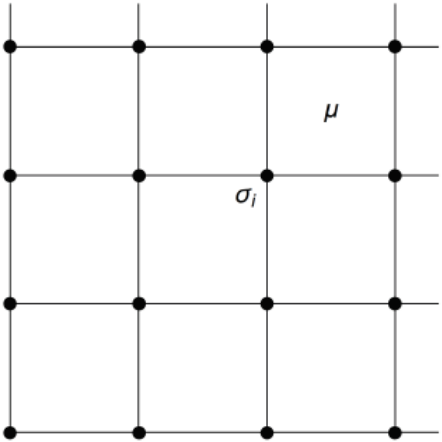
- 1st order phase transition:The two phases remain different at the transition. Order parameter discontinuous. The correlation length remains finite at this type of phase transition.
- 2nd order phase transition:The two phases become identical at the phase transition. All order parameters remain continuous. The correlation length diverges and the two phases become identical on either side as we approach the critical point. The system displays scale-invariance.

Critical exponents

Power-law dependencies of the system's correlation length and other thermodynamic quantities with respect to the parameters specifying distances from the critical point. The scaling hypothesis is used to relate critical exponents corresponding to various physical quantities.

Behaviour of systems near critical points is set into Universality Classes depending on the values critical exponents take.

Ising model in 2 dimensions



- Second order phase transition
- Spin variables $\sigma_i \in \{-1, 1\}$ sit at the nodes of a square lattice of size $N \times N$.
- Interaction through a nearest neighbor energy defined as: $E_{ij} = -J\sigma_i\sigma_j$
- Continuum Limit formulation: Separation of a low temperature ordered phase $K = \beta \cdot J > K_c$ with the expectation value $< \sigma >= 0$, from a high-temperature disordered phase with $K < K_c$ and $< \sigma >\neq 0$.
- Kramers-Wannier duality between σ and μ.

Connection with the Majorana Fermion

Majorana Fermion Lagrangian density: $\mathcal{L} = \overline{\psi}\partial\overline{\psi} + \psi\bar{\partial}\psi + im\psi\overline{\psi}$

- 1st step: Connection between the 2D Ising Model and the 1D Quantum Ising Chain.
- 2nd step: Jordan-Wigner Transformation = Pauli matrices written as annihilation-creation operators.
- 3rd step: Bogoliubov Transformation = Diagonalization of Hamiltonian for the normalization constant giving critical point.

No massive terms allowed in the Lagrangian in a CFT.

$$im\psi\overline{\psi} \propto (K - K_c)$$

At the critical point...

2D Ising model Lagrangian density: $\mathcal{L} = \overline{\psi}\partial\overline{\psi} + \psi\bar{\partial}\psi$

$M_{3,4}$ minimal model identification I - Conformal Families

The system's Conformal Families are:

$$[\Phi_{1,1}] = [\Phi_{2,3}] \quad : \quad (h_{1,1}, \bar{h}_{1,1}) = (0,0)$$

$$[\Phi_{2,1}] = [\Phi_{1,3}] \quad : \quad (h_{2,1}, \bar{h}_{2,1}) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

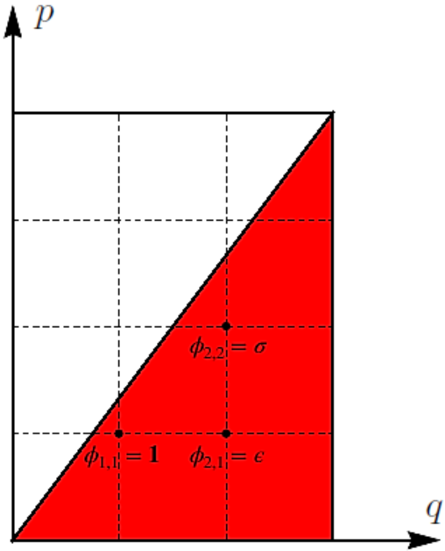
$$[\Phi_{1,2}] = [\Phi_{2,2}] \quad : \quad (h_{2,2}, \bar{h}_{2,2}) = \left(\tfrac{1}{16}, \tfrac{1}{16}\right)$$

Associated operator fields are given from:

- $I = \phi_{1,1}(z)\bar{\phi}_{1,1}(\bar{z}) \quad (-1)^{l_I = +1}$
- $\varepsilon(z,\bar{z}) = \phi_{2,1}(z)\bar{\phi}_{2,1}(\bar{z}) \quad (-1)^{l_\varepsilon} = +1$
- $\sigma(z,\bar{z}) = \phi_{1,2}(z)\bar{\phi}_{1,2}(\bar{z}) \quad (-1)^{l_\sigma} = -1$

$$\psi(z) = \phi_{(2,1)}(z)\bar{\phi}_{(1,1)}(\bar{z})$$

$$\bar{\psi}(\bar{z}) = \phi_{(1,1)}(z)\bar{\phi}_{(2,1)}(\bar{z})$$



$M_{3,4}$ minimal model identification II - Fusion Rules/ Critical Exponents

The allowed fusion rules are governed by spin reversal $(\sigma \to -\sigma)$ and the order-disorder duality $(\epsilon \to -\epsilon)$ symmetries:

Fusion Rules

- $[\sigma] \times [\sigma] = [I] + [\varepsilon]$
- $[\varepsilon] \times [\varepsilon] = [I]$
- $[\sigma] \times [\varepsilon] = [\sigma]$

$$\xi(T) \sim \frac{1}{|T - T_c|} \quad < \sigma_i \sigma_j >_c \sim \frac{1}{|r_i - r_j|^{\eta}}$$

$$< \varepsilon_i \varepsilon_j > \sim < \sigma_i \sigma_{i+1} \sigma_j \sigma_{j+1} > \sim \frac{1}{|r_i - r_j|^{2(2-1/\nu)}}$$

Critical Exponents

- Microscopic Critical Exponents: $\eta = 1/4$ $\nu = 1$
- Macroscopic Critical Exponents: $\alpha = 0$ $\beta = 1/8$ $\gamma = 7/4$ $\delta = 15$

Minimal Model $\mathcal{M}_{3,4}$ is the critical continuum version of the 2d Ising model.

2d Ising model Structure Constants I

We know

$$\left(\mathcal{L}_{-2} - \frac{4}{3}\mathcal{L}_{-1}^{2}\right) < \sigma(z,\bar{z})\phi(z_1\bar{z}_1)...\phi(z_N\bar{z}_N) >= 0$$

From the fusion rules and the 2-point correlation functions we can derive:

$$C_{\varepsilon\varepsilon\mathbf{I}} = C_{\sigma\sigma\mathbf{I}} = 1$$

For the final structure constant we have:

$$G^{(4)}_{\sigma\sigma\sigma\sigma} = \frac{1}{|x(1-x)|^{1/4}} \sum_{i,j=1}^{2} c_{ij} f_i(x) \bar{f}_j(\bar{x})$$

giving the ODE

$$\left[x(1-x)\frac{d^2}{dx^2} + \left(\frac{1}{2} - x\right)\frac{d}{dx} + \frac{1}{16}\right] f_i(x) = 0$$

with solution

$$f_1(x) = \sqrt{1+\sqrt{1-x}} \qquad f_2(x) = \sqrt{1-\sqrt{1-x}}$$

2d Ising model Structure Constants II

$$G^{(4)}_{\sigma\sigma\sigma\sigma} = \frac{C}{|x(1-x)|^{1/4}}\left(|1+\sqrt{1-x}| + |1-\sqrt{1-x}|\right) = \frac{C}{|x|^{1/4}}\left(2 + \frac{1}{2}|x| + ...\right)$$

Combining now $\quad G^{(4)}_{\sigma\sigma\sigma\sigma} = \lim_{z,\bar{z}\to\infty} |z|^{1/4} < \sigma(z,\bar{z})\sigma(1,1)\sigma(x,\bar{x})\sigma(0,0) >$

with

$$\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2) = \frac{1}{|z_1-z_2|^{1/4}} + \frac{C_{\sigma\sigma\varepsilon}}{|z_1-z_2|^{3/4}}\varepsilon(z_2,\bar{z}_2) + ...$$

we get

$$G^{(4)}_{\sigma\sigma\sigma\sigma} = \frac{1}{|x|^{1/4}}(1 + |x| C^2_{\sigma\sigma\varepsilon})$$

Comparing both expressions means

$$C = \frac{1}{2} \qquad\qquad C_{\sigma\sigma\varepsilon} = \frac{1}{2}$$

So the CFT we have found completely determines the dynamics of the system and the Ising model is solved by consistency.

Recent Developments...

· The AdS/CFT correspondence in studying theories of quantum gravity, formulated in terms of string theory or M-theory.
· 2d Boundary Conformal Field theory (bCFT) used now to study physics of open strings and D-branes.
· Again CFT is used to study other statistical models such as the Yang-Lee edge singularity which is a topic of intensive research the last couple of years.

Thank you for your attention!