Introduction to Vertex Algebras and their Representations

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Outline

- Calculus of formal distributions.
- Towards vertex algebras.
- Structure theory of vertex algebras.
- Representation theory of vertex (operator) algebras and construction of examples related to the Virasoro algebra.

Why vertex algebras?

- Vertex operator algebras are the mathematical foundation of conformal quantum field theories.
- Many formal manipulations in conformal field theory which are not always easy to justify become more accessible and true assertions for vertex algebras.
- Why CFTs then? Two-dimensional CFTs are the the best mathematically understood nontopological quantum field theories.
- Gaining more insight might help us shed light into more difficult problems such as the existence of four-dimensional quantum Yang-Mills theory and the mass gap problem.

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Formal Calculus

Formal distributions

- Let V be a vector space.
  A V*-valued formal distribution* (or *formal Laurent series*) in the indeterminate z is defined as

$$A(z) = \sum_{n\in\mathbb{Z}} v_n z^n \qquad v_n \in V$$

- These form a vector space, denoted as

$$V[[z, z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V \right\}$$

- In most cases V will be a space of operators acting on some vector space such as $\mathrm{End}(V)$. It can also be a super vector space or a Lie algebra.
- Is $V[[z, z^{-1}]]$ a ring? No. We have to be careful when multiplying two formal distributions.
- Products of two formal distributions will only make sense if the coefficient of each monomial in the product, acts as a finite sum of operators when applied to any vector in V.

$$A(z)\cdot B(z) = \sum_{n\in\mathbb{Z}} A_n z^n \cdot \sum_{m\in\mathbb{Z}} B_m z^m = \sum_{k\in\mathbb{Z}} \left(\sum_{n+m=k} A_n B_m \right) z^k$$

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Basics and notation

- Let

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^n \ \in V[[z, z^{-1}]]$$

  Define the *residue* as

$$\operatorname{Res}_z A(z) = A_{-1}$$

- The *formal derivative* operator acts on $V[[z, z^{-1}]]$ as

$$\partial_z A(z) = \sum_{n\in\mathbb{Z}} n A_n z^{n-1}$$

  and obeys the Leibniz rule.
- Conventional “Fourier expansion” notation:

$$A(z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$$

where $\quad A_{(n)} = A_{-n-1} = \operatorname{Res}_z \left(z^n A(z)\right)$

The notion of expansion

- Given a rational function $R(z,w)$ with poles only at $|z|=|w|$, $z=0$, $w=0$ we denote by
  1. $i_{z,w}R$ the power series expansion of R in the domain $|z|>|w|$.
  2. $i_{w,z}R$ the power series expansion of R in the domain $|z|<|w|$.
- These expansions can be derived from the following basic result which holds for any $k \in \mathbb{Z}$

$$i_{z,w}(z-w)^k = \sum_{j=0}^{\infty} \binom{k}{j} (-w)^j z^{k-j}$$

$$i_{w,z}(z-w)^k = \sum_{j=0}^{\infty} \binom{k}{j} z^j (-w)^{k-j}$$

- Using these expansions, one can define a *formal Taylor expansion* for any formal distribution $A(z)$ as

$$A(z) = \sum_{j=0}^{\infty} \frac{\partial_z^j A(z)|_{z=w}}{j!} (z-w)^j$$

valid in $|z-w|<|w|$.

- The *formal Dirac's distribution* $\delta(z,w)$ is defined by

$$\delta(z,w) = \sum_{n\in\mathbb{Z}} z^{-1}\left(\frac{w}{z}\right)^n = \sum_{n\in\mathbb{Z}} w^{-1}\left(\frac{z}{w}\right)^n \in \mathbb{C}[[z^{\pm}, w^{\pm}]]$$

or

$$\delta(z,w) = i_{z,w}\frac{1}{z-w} - i_{w,z}\frac{1}{z-w}$$

- It is the unique formal distribution satisfying

$$\mathrm{Res}_z\left(A(z)\delta(z,w)\right) = A(w) \quad \text{for any } A(z) \in V[[z^{\pm}]]$$

- Let $n, m \in \mathbb{N}$. The formal distribution $\delta(z,w)$ satisfies:
  1. $(z-w)^m \partial_w^n \delta(z,w) = 0 \qquad \forall m > n$
  2. $(z-w)\frac{1}{n!}\partial_w^n\delta(z,w) = \frac{1}{(n-1)!}\partial_w^{n-1}\delta(z,w)$ for $n \geq 1$
  3. $\delta(z,w) = \delta(w,z)$
  4. $\partial_z\delta(z,w) = -\partial_w\delta(w,z)$
  5. $a(z)\delta(z,w) = a(w)\delta(z,w)$, where $a(z)$ is any formal distribution.
  6. $\mathrm{Res}_z\, a(z)\delta(z,w) = a(w)$, which implies that $\delta(z,w)$ is unique.
  7. $e^{\lambda(z-w)}\partial_w^n\delta(z,w) = (\lambda + \partial_w)^n\,\delta(z,w)$

Locality

- Locality is the key notion behind the so-called Operator Product Expansions (OPEs) heavily used in CFT.
- A formal distribution $a(z,w) \in V[[z^{\pm}, w^{\pm}]]$ is called *local* if there exists $n \in \mathbb{N}$ such that

$$(z-w)^n a(z,w) = 0$$

- Dirac's $\delta(z,w)$ and its derivatives are local.
- In general, if $a(z,w)$ is local, then its derivatives are local.

Decomposition Theorem

Any local formal distribution $a(z,w) \in V[[z^{\pm}, w^{\pm}]]$ can be uniquely decomposed as the following finite sum:

$$a(z,w) = \sum_{j \in \mathbb{Z}^+} c^j(w) \frac{\partial_w^j \delta(z,w)}{j!}$$

where $c^j(w) \in V[[w^{\pm}]]$ are given by

$$c^j(w) = \mathrm{Res}_z (z-w)^j a(z,w)$$

The converse statement is also true.

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Locality of two formal distributions

- To make things more interesting, suppose V has the additional structure of a Lie algebra and let us denote by $\mathfrak{g}$.
- Then define a new formal distribution called the *bracket* of two formal distributions:

$$[a(z), b(w)] = \sum_{m,n \in \mathbb{Z}} [a_{(m)}, b_{(n)}] z^{-1-m} w^{-1-n} \ \in \mathfrak{g}[[z^{\pm}, w^{\pm}]]$$

- Careful! The bracket on the right hand side is the Lie bracket in $\mathfrak{g}$.
- Two formal distributions $a(z), b(z) \in \mathfrak{g}[[z^{\pm}]]$ are *local with respect to each other* if there exists $n \in \mathbb{N}$ such that

$$(z-w)^n [a(z), b(w)] = 0$$

- The last condition actually originates from one of Wightman's axioms, namely the axiom of local commutativity or microcausality.
- Applying the decomposition theorem in this case, we have

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Normal ordering

- Let us partition a formal distribution into

$$a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-n-1}, \qquad a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}$$

- Define the *normally ordered product* as the following formal distribution

$$: a(z)b(w) := a(z)_+ b(w) + b(w)a(z)_-$$

with values in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Big locality theorem

Theorem

Each of the following equations is equivalent to the condition of locality:

1. $$[a(z), b(w)] = \sum_{j=0}^{N-1} \frac{c^j(w)}{j!} \partial_w^j \delta(z,w) \quad \text{with} \quad c^j(w) = \mathrm{Res}_z (z-w)^j [a(z), b(w)]$$

2. $$a(z)b(w) = \sum_{j=0}^{N-1} \left(i_{z,w} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + :a(z)b(w):$$
$$b(w)a(z) = \sum_{j=0}^{N-1} \left(i_{w,z} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + :a(z)b(w):$$

3. $$\left[a_{(m)}, b_{(n)}\right] = \sum_{j=0}^{N-1} \begin{pmatrix} m \\ j \end{pmatrix} c^j_{(m+n-j)}, \quad m,n \in \mathbb{Z}$$

4. $$\left[a_{(m)}, b(w)\right] = \sum_{j=0}^{N-1} \begin{pmatrix} m \\ j \end{pmatrix} c^j(w) w^{m-j}, \quad m \in \mathbb{Z}$$

Operator Product Expansion

- By abuse of notation, we sometimes write the first equation of 2. as

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}} + :a(z)b(w):$$

  or write just the singular part

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}$$

- These product relations are starting to look familiar. They are called *Operator Product Expansions* (OPEs) and are heavily encountered in CFT.

Fourier transform

- We define the *Fourier transform* in two indeterminates of $a(z,w) \in \mathfrak{g}[[z^{\pm}, w^{\pm}]]$ by

$$F_{z,w}^{\lambda} a(z,w) = \mathrm{Res}_z \, e^{\lambda(z-w)} a(z,w)$$

- With the following useful properties:
  1. If $a(z,w)$ is local, then $F_{z,w}^{\lambda} a(z,w) \in \mathfrak{g}\,[[w^{\pm}]]\,[\lambda]$
  2. $F_{z,w}^{\lambda} \partial_z a(z,w) = -\lambda F_{z,w}^{\lambda} a(z,w) = \left[\partial_w, F_{z,w}^{\lambda}\right] a(z,w)$
  3. If $a(z,w)$ is local, then $F_{z,w}^{\lambda} a(w,z) = F_{z,w}^{-\lambda-\partial_w} a(z,w)$
  4. $F_{z,w}^{\lambda} F_{x,w}^{\mu} a(z,w,x) = F_{x,w}^{\lambda+\mu} F_{z,x}^{\lambda} a(z,w,x)$
- Using the aforementioned OPE we have:

$$F_{z,w}^{\lambda} \left(a(z,w)\right) = \sum_{j \in \mathbb{Z}^+} \frac{\lambda^j}{j!} c^j(w), \qquad c^j(w) = \mathrm{Res}_z (z-w)^j a(z,w)$$

This is a generating series for the OPE coefficients of a local formal distribution.

Towards Vertex Algebras

j-products and λ-bracket

- Let $a, b \in \mathfrak{g}[[z^{\pm}]]$. Define the following compact notation for the OPE coefficients

$$a_{(j)}b \equiv a(w)_{(j)}b(w) = \mathrm{Res}_z (z-w)^j [a(z), b(w)]$$

  called the *j-product* of $a(z)$ and $b(z)$.
- Their *λ-bracket* is defined as

$$[a_\lambda b] \equiv [a(w)_\lambda b(w)] = F^\lambda_{z,w}[a(z), b(w)]$$

- Notice

$$[a_\lambda b] = \sum_{j \in \mathbb{Z}^+} \frac{\lambda^j}{j!} \left(a_{(j)} b \right)$$

- Properties:
  1. $(\partial a)_{(j)} b = -j a_{(j-1)} b$
  2. $a_{(j)} \partial b = \partial \left(a_{(j)} b \right) + j a_{(j-1)} b$
  3. $\partial (a_{(j)} b) = (\partial a)_{(j)} b + a_{(j)} \partial b$
  4. $\partial [a_\lambda b] = [\partial a_\lambda b] + [a_\lambda \partial b]$
  5. $[\partial a_\lambda b] = -\lambda \, [a_\lambda b]$
  6. $[a_\lambda \partial b] = (\partial + \lambda) \, [a_\lambda b]$
  7. $[b_\lambda a] = -[a_{-\lambda-\partial} b]$ if (a, b) is a local pair.
  8. $[a_\lambda \, [b_\mu c]] = \left[[a_\lambda b]_{\lambda+\mu} \, c \right] + [b_\mu \, [a_\lambda c]]$

Example: Virasoro algebra

- The Virasoro algebra $\mathfrak{V}$ is an infinite dimensional complex Lie algebra. A basis is given by the set $\{L_m, C; m \in \mathbb{Z}\}$ with the following bracket relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{C}{12}(m^3 - m)$$
$$[L_m, C] = 0, \qquad \forall m, n \in \mathbb{Z}$$

- Starting from $\mathfrak{V}$ we construct $\mathfrak{V}$-valued formal distributions by setting conventionally

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad L_n \in \mathfrak{V}$$

called *Virasoro formal distributions*.

- Then the aforementioned commutation relations are equivalent to

$$[L(z), L(w)] = \partial_w L(w)\delta(z,w) + 2L(w)\partial_w\delta(z,w) + \frac{C}{12}\partial_w^3\delta(z,w)$$

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Example: Virasoro algebra

- Since $[L(z), C] = 0$, we say that $\mathcal{F} = \{L(z), C\}$ is a *local family* of $\mathfrak{V}$-valued formal distributions.
- In terms of the λ-bracket, the commutation relations are equivalent to

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12}C$$

- and in terms of j-products, they are equivalent to

$$L_{(0)}L = \partial L, \quad L_{(1)}L = 2L, \quad L_{(3)}L = \frac{C}{2}, \quad L_{(j)}L = 0 \;\; \forall j \neq \{0, 1, 3\}$$

- We call the pair $(\mathfrak{V}, \mathcal{F})$ a *formal distributions Lie algebra*.

Fields and the generalized j-product

- Why? The factor $:a(z)b(w):$ appearing in the OPE is not well-defined in the limit $z \to w$.
- A *(quantum) field* $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ is an $\operatorname{End} V$-valued formal distribution such that

$$\forall v \in V \quad \exists N \in \mathbb{N} : \quad a_{(n)}v = 0 \quad \forall n \geq N$$

- The normal product of two fields is again a field $\implies$ the subspace of fields in $\operatorname{End} V[[z^{\pm}]]$ is an algebra with respect to the normal product.
- We can now define j-products for negative j:

$$a(w)_{(-j-1)}b(w) = :\frac{\left(\partial_w^j a(w)\right) b(w)}{j!}: \qquad j \in \mathbb{Z}^+$$

$$a(w)_{(j)}b(w) = \operatorname{Res}_z (z-w)^j [a(z), b(w)] \qquad j \in \mathbb{Z}^+$$

- *Generalized j-product* for $j \in \mathbb{Z}$:

$$a(w)_{(j)}b(w) = \operatorname{Res}_z \left(i_{z,w}(z-w)^j a(z)b(w) - i_{w,z}(z-w)^j b(w)a(z) \right)$$

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Lemma

If $a(w), b(w), c(w)$ are mutually local formal distributions, then $(a, b_{(j)}c)$ is a local pair for all $j \in \mathbb{Z}$.

- General Wick's formula:

$$\left[a_\lambda \left(b_{(n)}c\right)\right] = \sum_{k\in\mathbb{Z}^+} \frac{\lambda^k}{k!} [a_\lambda b]_{(n+k)} c + b_{(n)} [a_\lambda c]$$

or

$$a(w)_{(m)}(b(w)_{(n)}c(w)) = \sum_{j=0}^{m} \binom{m}{j} (a(w)_{(j)}b(w))_{(m+n-j)}c(w) + b(w)_{(n)}(a(w)_{(m)}c(w))$$

- *Non-abelian Wick's formula*:

$$[a_\lambda : bc :] =: [a_\lambda b]\, c : + : b\, [a_\lambda c] : + \int_0^\lambda \left[[a_\lambda b]_\mu\, c\right] d\mu$$

Structure Theory of Vertex Algebras

"*I am an old man, and I know that a definition cannot be so complicated.*"

I.M Gelfand (after a talk on vertex algebras in his Rutgers seminar)

A *vertex algebra* $(V, |0\rangle, Y, T)$ is the following data:

1. V is a vector space called the *space of states*.
2. $|0\rangle \in V$ is called the *vacuum vector*.
3. Y is a linear map from V to the space of $\mathrm{End}\, V$-valued **fields**

$$Y(\cdot, z) : V \to \mathrm{End}\, V[[z, z^{-1}]]$$
$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

   which is said to realize the *state-field correspondence*. The fields $Y(a, z)$ are called *vertex operators*.

4. $T \in \mathrm{End}\, V$ is a linear map called the *infinitesimal translation operator*.

Such that the following axioms hold

(A1) ***Vacuum***:

$$Y(|0\rangle, z) = I_V$$
$$Y(a,z)|0\rangle\Big|_{z=0} = a \qquad (\textit{\textbf{Creation axiom}})$$
$$T|0\rangle = 0$$

(A2) ***Translation covariance***:

$$[T, Y(a,z)] = \partial_z Y(a,z)$$

(A3) ***Locality***:

$\{Y(a,z) \,|\, a \in V\}$ is a local family of fields, i.e. for all $a, b \in V$ we have

$$(z-w)^n [Y(a,z), Y(b,w)] = 0 \qquad \text{for some } n \in \mathbb{N}$$

Rewriting the axioms

(A1) ***Vacuum***:

$$|0\rangle_{(n)}\, a = \delta_{n,-1} a \qquad n \in \mathbb{Z}$$

$$a_{(n)}\, |0\rangle = \delta_{n,-1} a \qquad n \geq -1 \quad (\textbf{\textit{Creation axiom}})$$

(A2) ***Translation covariance***:

$$[T, a_{(n)}] = -n a_{(n-1)} \qquad n \in \mathbb{Z}$$

- Notice that these axioms imply

$$T(a) = a_{(-2)}|0\rangle$$

- A local family of fields $F \subset \operatorname{End} V[[z^{\pm}]]$ can also be endowed with a vertex algebra structure.

Uniqueness and n-products theorems

Uniqueness Theorem

Let V be a vertex algebra and $B(z)$ be an $\mathrm{End}\, V$-valued field such that:

(1) $(B(z), Y(a,z))$ is a local pair for all $a \in V$.

(2) $B(z)\,|0\rangle = Y(b,z)\,|0\rangle$

Then we have $B(z) = Y(b,z)$.

n-products Theorem

Let V be a vertex algebra. Then the following identity holds

$$Y(a_{(n)}b, z) = Y(a,z)_{(n)}Y(b,z)$$

Corollary

1. $Y(a_{(-1)}b, z) =: Y(a,z)Y(b,z):$
2. $:\frac{\partial^n Y(a,z)}{n!}Y(b,z) := Y(a_{(-n-1)}b, z)$
3. $Y(Ta, z) = \partial_z Y(a,z)$

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Then we have $B(z) = Y(b,z)$.

n-products Theorem

Let V be a vertex algebra. Then the following identity holds

$$Y(a_{(n)}b, z) = Y(a,z)_{(n)}Y(b,z)$$

Corollary

1. $Y(a_{(-1)}b, z) =: Y(a,z)Y(b,z) :$
2. $: \frac{\partial^n Y(a,z)}{n!} Y(b,z) := Y(a_{(-n-1)}b, z)$
3. $Y(Ta, z) = \partial_z Y(a,z)$

First fundamental theorem

Existence Theorem

Let V be a vector space, $|0\rangle \in V$ and $T \in \operatorname{End} V$. Let $\mathcal{F} = \{a^j(z) = \sum_n a^j_{(n)} z^{-1-n}\}_{j \in J}$ be a collection of $\operatorname{End} V$-valued fields such that the following properties hold:

(1) $a^j(z)\,|0\rangle\,\big|_{z=0} = a^j \in V$ and $T\,|0\rangle = 0$.

(2) $[T, a^j(z)] = \partial_z a^j(z)$

(3) All pairs $(a^i(z), a^j(w))$ are local.

(4) The vectors $a^{j_s}_{(n_s)} \ldots a^{j_1}_{(n_1)}\,|0\rangle$ span V.

Then the formula

$$Y\left(a^{j_s}_{(n_s)} \cdots a^{j_1}_{(n_1)}\,|0\rangle\,, z\right) = a^{j_s}(z)_{(n_s)}(\cdots a^{j_2}(z)_{(n_2)}(a^{j_1}(z)_{(n_1)} I_V)\cdots)$$

defines a unique structure of a vertex algebra on V with vacuum vector $|0\rangle$, infinitesimal translation operator T, such that

$$Y(a^j, z) = a^j(z)$$

- Let V be a vertex algebra. Then the following *Borcherds' identity* holds for all $n \in \mathbb{Z}$ and $a, b \in V$.

$$Y(a,z)Y(b,w)i_{z,w}(z-w)^n - Y(b,w)Y(a,z)i_{w,z}(z-w)^n = \sum_{j \in \mathbb{Z}^+} Y\left(a_{(n+j)}b, w\right) \frac{\partial_w^j \delta(z,w)}{j!}$$

- Special $n = 0$ case leads to *Borcherds' commutator formula*:

$$[Y(a,z), Y(b,w)] = \sum_{j \in \mathbb{Z}^+} Y(a_{(j)}b, w) \frac{\partial_w^j \delta(z,w)}{j!}$$

- *skew-symmetry* :

$$Y(a,z)b = e^{zT} Y(b,-z)a$$

Borcherds' identity and skew-symmetry

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- A *conformal vector* of a vertex algebra V is a vector ω such that the corresponding field $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is a Virasoro field with central charge c which has the following properties
  1. $L_{-1} = T$
  2. L_0 is diagonalizable on V. Its eigenvalues are called *conformal weights*.
  3. $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{C}{12}(m^3 - m), \qquad [L_m, C] = 0 \quad \forall m, n \in \mathbb{Z}$
- A *vertex operator algebra* $(V, Y, |0\rangle, \omega)$ is a $\mathbb{Z}$-graded vector space (graded by conformal weights)

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)} \qquad \text{where} \quad L_0 v = nv \qquad \text{for } v \in V_{(n)}$$

such that the following grading restrictions hold:

$$\dim V_{(n)} < \infty \qquad \text{for } n \in \mathbb{Z}$$

$$V_{(n)} = 0 \qquad \text{for } n \text{ sufficiently negative}$$

equipped with a vertex algebra structure $(V, |0\rangle, Y)$ and a distinguished conformal vector ω.

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Properties

- $$L_n V_{(j)} \subset V_{(j-n)} \qquad \text{and} \qquad T V_{(j)} \subset V_{(j+1)}$$
- $$\begin{aligned} [L_{-1}, Y(v,z)] &= \partial_z Y(v,z) = Y(L_{-1}v, z) \\ [L_0, Y(v,z)] &= Y(L_0 v, z) + zY(L_{-1}v, z) \\ [L_1, Y(v,z)] &= Y(L_1 v, z) + 2zY(L_0 v, z) + z^2 Y(L_{-1}v, z) \\ L_n \left|0\right\rangle &= \delta_{n,-2}\, \omega \quad \text{for } n \geq -2 \end{aligned}$$
- $$\mathrm{wt} \left|0\right\rangle = 0 \qquad \text{and} \qquad \mathrm{wt}\, \omega = 2$$
- One can define the notions of subalgebras, ideals, homomorphisms of vertex (operator) algebras, similarly to the “classical” case.

Representation Theory of Vertex Algebras

- Let $(V, |0\rangle, T, Y)$ be a vertex algebra. A vector space W is called a *V-module (for V viewed as a vertex algebra)* if it is equipped with a linear map

$$\begin{aligned} Y_W : V &\to \operatorname{End} W[[z, z^{-1}]] \\ A &\mapsto Y_W(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)}^W z^{-n-1} \end{aligned}$$

which assigns to each $A \in V$ a **field**, such that

1. $Y_W(|0\rangle, z) = I_W$.
2. $Y_W(A_{(n)}B, z) = Y_W(A, z)_{(n)} Y_W(B, z)$ for $A, B \in V$.
3. $\{Y_W(A, z)\}_{A \in V}$ is a local family.

- Let V be a vertex operator algebra. A *V-module* is a module W for V viewed as a vertex algebra such that

$$W = \bigoplus_{h \in \mathbb{C}} W_{(h)} \qquad \text{where} \quad W_{(h)} = \{w \in W \mid L_0^W w = hw\}$$

is the subspace of W of vectors of conformal weight h, and such that the following *grading restrictions* hold:

$$\begin{aligned} &\dim W_{(h)} < \infty \\ &W_{(h)} = 0 \quad \text{for } h \text{ whose real part is sufficiently negative} \end{aligned}$$

Modules over vertex algebras

- Let $(V, |0\rangle, T, Y)$ be a vertex algebra. A vector space W is called a V*-module* (*for* V *viewed as a vertex algebra)* if it is equipped with a linear map

$$\begin{aligned} Y_W : V &\to \operatorname{End} W[[z, z^{-1}]] \\ A &\mapsto Y_W(A, z) = \sum_{n \in \mathbb{Z}} A^W_{(n)} z^{-n-1} \end{aligned}$$

  which assigns to each $A \in V$ a **field**, such that
  1. $Y_W(|0\rangle, z) = I_W$.
  2. $Y_W(A_{(n)}B, z) = Y_W(A, z)_{(n)} Y_W(B, z)$ for $A, B \in V$.
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Vertex Operator Algebras and modules associated to the Virasoro algebra

Setup

- Simplify notation: $Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ and $|0\rangle \equiv \mathbf{1}$
- Recall that the Virasoro algebra $\mathfrak{V}$ is the Lie algebra with basis $\{L_m, C \mid m \in \mathbb{Z}\}$ with

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C, \qquad [L_m, C] = 0$$

- Recall the creation axiom of the vertex algebra:

$$L_n\mathbf{1} = 0, \quad n \geq -1$$

- A $\mathfrak{V}$-module W is called a *restricted module* if for any $w \in W$ we have $L_n w = 0$ for n sufficiently large.

Setup

- It is a $\mathbb{Z}$-graded Lie algebra: $\mathfrak{V} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{V}_{(n)}$, where

$$\mathfrak{V}_{(0)} = \mathbb{C}L_0 \oplus \mathbb{C}C \quad \text{and} \quad \mathfrak{V}_{(n)} = \mathbb{C}L_{-n} \quad \text{for } n \neq 0$$

  and the grading is given by ad_{L_0}-eigenvalues called *conformal weights*.
- It has the following graded subalgebras:
  1. $\mathfrak{V}_{(\pm)} = \bigoplus_{n\geq 1} \mathfrak{V}_{(\pm n)} = \bigoplus_{n\geq 1} \mathbb{C}L_{\mp n}$
  2. $\mathfrak{V}_{(0)} \oplus \mathfrak{V}_{(-)}$ and $\mathfrak{V}_{(0)} \oplus \mathfrak{V}_{(+)}$
  3. $\mathfrak{V}_{\leq 1} = \bigoplus_{n\leq 1} \mathfrak{V}_{(n)} = \mathfrak{V}_{(-)} \oplus \mathbb{C}L_0 \oplus \mathbb{C}C \oplus \mathbb{C}L_{-1}$
  4. $\mathfrak{V}_{\geq 2} = \bigoplus_{n\geq 2} \mathfrak{V}_{(n)} = \bigoplus_{n\geq 2} \mathbb{C}L_{-n}$
- We have the decomposition

$$\mathfrak{V} = \mathfrak{V}_{\leq 1} \oplus \mathfrak{V}_{\geq 2}$$

- The Virasoro algebra also admits the triangular decomposition

$$\mathfrak{V} = \mathfrak{V}_{(-)} \oplus \mathbb{C}L_0 \oplus \mathbb{C}C \oplus \mathfrak{V}_{(+)}$$

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First step: Construct a vertex operator algebra

- $\mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}C$ is a Cartan subalgebra of the Virasoro algebra. We also have another subalgebra $\mathfrak{b} = \mathfrak{V}_{(-)} \oplus \mathfrak{h}$, often called the Borel subalgebra.
- All representations can be labeled by the eigenvalues of L_0 and C on the representation space. Since we are interested in lowest weight representations, these will be labeled by the lowest weight denoted by h and the central charge c.
- Consider $\mathbb{C}$ as an $\mathfrak{V}_{\leq 1}$-module with C acting as the scalar c and with $\mathfrak{V}_{(-)} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_{-1}$ acting trivially. Denote this $\mathfrak{V}_{\leq 1}$-module by $\mathbb{C}_c$. Form the induced module

$$V_{\mathfrak{V}}(c, 0) = \mathcal{U}(\mathfrak{V}) \otimes_{\mathcal{U}(\mathfrak{V}_{\leq 1})} \mathbb{C}_c$$

- This is the Verma module of lowest weight 0 with lowest-weight vector $\mathbf{1}$ which is actually the image of $1 \otimes 1 \in \mathcal{U}(\mathfrak{V}) \otimes \mathbb{C}_c$ in $V_{\mathfrak{V}}(c, 0)$.
- From the Poincaré-Birkhoff-Witt theorem we have the following vector space isomorphism

$$V_{\mathfrak{V}}(c, 0) \cong \mathcal{U}(\mathfrak{V}_{\geq 2})$$

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- Then we have

$$V_{\mathfrak{Vir}}(c,0) = \bigoplus_{n\geq 0} V_{\mathfrak{Vir}}(c,0)_{(n)}$$

where $V_{\mathfrak{Vir}}(c,0)_{(n)}$ has a basis consisting of vectors

$$L_{-m_1}\cdots L_{-m_r}\mathbf{1}, \qquad r\geq 0,\ m_1\geq m_2\geq\cdots\geq m_r\geq 2 \text{ with } m_1+\cdots+m_r=n$$

- $V_{\mathfrak{Vir}}(c,0)$ is a restricted module in the sense that for any $v\in V_{\mathfrak{Vir}}(c,0)$ we have $L_n v=0$ for large enough n. It is generated by $\mathbf{1}$ with $L_n\mathbf{1}=0$ for $n\geq -1$.
- The two grading restrictions

$$\dim V_{\mathfrak{Vir}}(c,0)_{(n)} < \infty$$

$$V_{\mathfrak{Vir}}(c,0)_{(n)} = 0 \quad \text{for } n \text{ whose real part is sufficiently negative}$$

hold for $V_{\mathfrak{Vir}}(c,0)$.
- As a module for the Virasoro algebra, $V_{\mathfrak{Vir}}(c,0)$ is **universal**, i.e., for any module W of the Virasoro algebra of central charge c equipped with a vector $e\in W$ such that $L_n e=0$ for $n\geq -1$, there exists a unique module homomorphism from $V_{\mathfrak{Vir}}(c,0)$ to W sending $\mathbf{1}$ to e.

First step: Construct a vertex operator algebra

- Then we have
$$V_{\mathfrak{V}}(c,0)=\bigoplus_{n\geq 0}V_{\mathfrak{V}}(c,0)_{(n)}$$
where $V_{\mathfrak{V}}(c,0)_{(n)}$ has a basis consisting of vectors
$$L_{-m_1}\cdots L_{-m_r}\mathbf{1},\qquad r\geq 0,\ m_1\geq m_2\geq\cdots\geq m_r\geq 2\ \text{ with }\ m_1+\cdots+m_r=n$$

- $V_{\mathfrak{V}}(c,0)$ is a restricted module in the sense that for any $v\in V_{\mathfrak{V}}(c,0)$ we have $L_n v=0$ for large enough n. It is generated by $\mathbf{1}$ with $L_n\mathbf{1}=0$ for $n\geq -1$.
- The two grading restrictions
$$\dim V_{\mathfrak{V}}(c,0)_{(n)}<\infty$$
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First step: Construct a vertex operator algebra

Theorem

Let $c \in \mathbb{C}$ and let V be a module for $\mathfrak{V}$ of central charge c equipped with a vector $\mathbf{1}$ such that V is generated by $\mathbf{1}$ and $L_n\mathbf{1} = 0,\ n \geq -1$. Then, there exists a unique vertex operator algebra structure $(V, Y, \mathbf{1}, \omega)$ with $\mathbf{1}$ as the vacuum vector and $\omega = L_{-2}\mathbf{1}$ as the conformal vector such that

$$Y(\omega, z) = L_V(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

The map Y is given by the existence theorem as

$$Y(L_{n_1} \cdots L_{n_r}\mathbf{1}, z) = L_V(z)_{n_1+1} \cdots L_V(z)_{n_r+1} I_V$$

with $r \geq 0,\ n_1, \ldots, n_r \in \mathbb{Z}$. In particular this holds for $V_{\mathfrak{V}}(c, 0)$.

- Since $V_{\mathfrak{V}}(c, 0)$ is generated by $\omega = L_{-2}\mathbf{1}$, it is a **minimal** vertex operator algebra in the sense that it does not have any proper vertex operator subalgebra with the same conformal vector.

Second step: $\{\text{VOA-modules}\} \overset{!}{\longleftrightarrow} \{\mathfrak{V} - \text{modules}\}$

Theorem

1. Every module for $V_{\mathfrak{V}}(c,0)$ viewed as a vertex algebra is naturally a restricted module for the Virasoro algebra of central charge c, with $L_W(z) = Y_W(L_{-2}\mathbf{1}, z)$.
2. Conversely, every restricted module W for the Virasoro algebra of central charge c is naturally a module for $V_{\mathfrak{V}}(c,0)$ viewed as a vertex algebra with

$$Y_W\left(L_{n_1}\cdots L_{n_r}\mathbf{1}, z\right) = L_W(z)_{n_1+1}\cdots L_W(z)_{n_r+1} I_W$$

   for $r \geq 0,\ n_j \in \mathbb{Z}$.
3. The modules for $V_{\mathfrak{V}}(c,0)$ viewed as a vertex operator algebra are exactly those restricted modules for the Virasoro algebra of central charge c that are $\mathbb{C}$-graded by L_0-eigenvalues and with the two grading restrictions.
4. For any $V_{\mathfrak{V}}(c,0)$-module W, the $V_{\mathfrak{V}}(c,0)$-submodules of W are exactly the submodules of W for the Virasoro algebra and these submodules are also graded.

Third step: Construct a family of VOA-modules

- Generalize construction $\rightarrow$ get a family of restricted $\mathfrak{V}$-modules of central charge c that are $\mathbb{C}$-graded by L_0-eigenvalues and satisfy the two grading restrictions. These will be naturally modules for the VOA $V_{\mathfrak{V}}(c,0)$ by last theorem.
- Let $c,h \in \mathbb{C}$. Consider $\mathbb{C}$ as a $\mathfrak{V}_{(0)}$-module with C and L_0 acting as the scalars c and h respectively. Let $\mathfrak{V}_{(-)}$ act trivially on $\mathbb{C}$, making $\mathbb{C}$ a $\mathfrak{b}$-module, which we denote by $\mathbb{C}_{c,h}$. Form the induced module

$$M_{\mathfrak{V}}(c,h) = \mathcal{U}(\mathfrak{V}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{c,h}$$

- From PBW theorem, we have the following vector space isomorphism

$$M_{\mathfrak{V}}(c,h) \cong \mathcal{U}(\mathfrak{V}_{(+)})$$

- Denote the lowest-weight vector as $\mathbf{1}_{c,h} \in M_{\mathfrak{V}}(c,h)$ which is actually the image of $\mathbf{1} \otimes 1 \in \mathcal{U}(\mathfrak{V}) \otimes \mathbb{C}_{c,h}$ in $M_{\mathfrak{V}}(c,h)$.
- We have the following grading

$$M_{\mathfrak{V}}(c,h) = \bigoplus_{n\geq 0} M_{\mathfrak{V}}(c,h)_{(n+h)}$$

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- Where $M_{\mathfrak{V}}(c,h)_{(n+h)}$ is the L_0-eigenspace of eigenvalue $n+h$ and has a basis consisting of vectors

$$L_{-m_1}\cdots L_{-m_r}\mathbf{1}_{c,h}, \qquad r\geq 0,\ m_1\geq m_2\geq\cdots\geq m_r\geq 1$$
$$\text{with } m_1+\cdots+m_r=n$$

Theorem

For any complex numbers c, h, $W = M_{\mathfrak{V}}(c,h)$ has a unique module structure for the vertex operator algebra $V_{\mathfrak{V}}(c,0)$ such that $Y_W(\omega,z) = L_W(z)$.

- $M(c,h)$ is universal in the sense that for any module W for the Virasoro algebra of central charge c equipped with a vector w such that $L_0 w = hw$ and $L_n w = 0$ for $n\geq 1$, there exists a unique Virasoro module homomorphism from $M(c,h)$ to W sending $\mathbf{1}_{c,h}$ to w.
- It is not irreducible.

Final step: Construct irreducible modules

- Define the subspace $U(c,h)$ as the set of all vectors $v \in M(c,h)$ with $\mathbf{1}_{c,h}$-component equal to zero and such that the $\mathbf{1}_{c,h}$-component of

$$L_{n_1} \cdots L_{n_j} v \qquad n_1, \dots, n_j > 0$$

is also zero for any collection of $L_{n_1}, \dots, L_{n_j} \in \mathfrak{V}_{(-)}$. In other words, a vector v belongs to $U(c,h)$ if we cannot "get to" $\mathbf{1}_{c,h}$ from v by applying lowering operators $L_n,\ n > 0$.

Theorem

Let $M(c,h)$ be the Verma module for the Virasoro algebra with lowest weight h constructed above. Then

1. $U(c,h)$ is the largest proper invariant subspace of $M(c,h)$ with respect to the Virasoro algebra action.
2. The quotient space $L(c,h) := M(c,h)/U(c,h)$ is an irreducible lowest weight cyclic Virasoro-module with lowest weight h and lowest weight vector being the image of v_0 in the quotient $L(c,h)$.

Final step: Construct irreducible modules

Theorem

For any complex numbers c, h

1. $L(c,h)$ is an irreducible module for the vertex operator algebra $V(c,0)$.
2. The modules $L(c,h)$ for $h \in \mathbb{C}$ exhaust the irreducible $V(c,0)$-modules up to equivalence.

- $L(c,0)$ is the unique simple minimal vertex operator algebra of central charge c.
- $V(c,0)$ is an irreducible Virasoro-module if and only if $c \neq c_{p,q}$ where

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$$

  for $p, q \in \{2,3,4,...\}$ with p and q relatively prime. In this case $V(c,0) = L(c,0)$ and all irreducible $L(c,0)$-modules are classified by last theorem. On the other hand if $c = c_{p,q}$, every irreducible $L(c,0)$-module is isomorphic to $L(c,h)$ for some $h \in \mathbb{C}$.
- $L(c_{p,q},0)$ is a rational vertex algebra, whose irreducible modules form the "minimal model" of conformal field theory defined by Belavin, Polyakov and Zamolodchikov.

What have we done?

- We have seen that certain vector spaces (related to “physics”) can be endowed with the extra natural structure of a vertex algebra.

What is next?

- First, give geometric meaning.
- Construct **full**, rational CFTs satisfying the axioms of Kontsevich- Segal and Moore-Seiberg. In particular, Wess-Zumino-Witten models and minimal models.
- More specifically, construct higher-genus correlation functions.
- Construct and study logarithmic conformal field theories.